Lecture 9 - Triple integrals in cartesian coordinates - 7/23/2014 — Interphase 2014 Calc 3

24.) Triple integrals (conceptually)

a. The triple integral as a Riemann sum:

$$\iiint_D f(x,y,z)dA \approx \sum f(x,y,z)\Delta V$$

where the sum is over all small cubes of volume ΔV that are part of D.

b. To write down a triple integral for a quantity, consider a small cube of volume ΔV located at (x,y,z). This cube will contribute $f(x,y,z)\Delta V$ to the quantity of interest. The total quantity is then given by the triple integral $\iiint_D f(x,y,z)dV$.

25. Applications of triple integrals

a. The volume of a 3d region D is $\iiint_D dV$.

b. The mass of a 3d region D with density $\rho(x,y,z)$ is $M = \iiint_D \rho(x,y,z)dV$.

c. The moment of inertia of a 3d object with density $\rho(x,y,z)$ is $I = \iiint_D \rho(x,y,z)d^2(x,y,z)dV$, where $d(x,y,z)$ is the distance from (x,y,z) to the axis of rotation.

e. The average value of a function $f(x,y,z)$ over the 3d region D is

$$\bar{f} = \frac{\iiint_D f(x,y,z)dV}{\iint_D dV}$$

d. The coordinates of the center of mass of a 3d object D with density $\rho(x,y,z)$ is

$$\bar{x} = \frac{\iiint_D x\rho(x,y,z)dV}{\iiint_D \rho(x,y,z)dV} \text{ and } \bar{y} = \frac{\iiint_D y\rho(x,y,z)dV}{\iiint_D \rho(x,y,z)dV} \text{ and } \bar{z} = \frac{\iiint_D z\rho(x,y,z)dV}{\iiint_D \rho(x,y,z)dV}$$

26. Evaluating triple integrals in Cartesian coordinates

a. In Cartesian coordinates, $dV = dxdydz$.

b. The double integral over the rectangular prism $[a,b] \times [c,d] \times [e,f]$ can be evaluated as

$$\iint_{[a,b]\times[c,d]\times[e,f]} f(x,y,z)dV = \int_a^b \left(\int_c^d \left(\int_e^f f(x,y,z)dz \right) dy \right) dx$$

c. Many complicated regions can be written like:

$$x_{\min} \leq x \leq x_{\max},\ y_{\min}(x) \leq y \leq y_{\max}(x),\ z_{\min}(x, y) \leq z \leq z_{\max}(x, y)$$

d. If D is of the form above, then the triple integral can be written as an iterated integral:

$$\iint_D f(x, y, z)dxdydz = \int_{x_{\min}}^{x_{\max}} \left(\int_{y_{\min}(x)}^{y_{\max}(x)} \left(\int_{z_{\min}(x,y)}^{z_{\max}(x,y)} \right) f(x, y)dy \right) dx$$

e. Some regions may be easier to describe with y or z as the out outer variable of integration. Choose the order of variables by specifying the region in the way that is the simplest.