Lecture 5

7/11/2014

Gradient

Directional derivative

Chain Rule + Total Derivative

Warmup

Plot level set of value 1 of

a) $f(x,y) = y^3 - x$

b) $g(x,y,z) = \frac{x^2}{4} + y^2 + z^2$

a) $$\underbrace{\vec{\nabla f}}_{\substack{\text{gradient}\\ \text{of}\\ \text{a function}\\ \text{is a vector}}}\underbrace{(x,y)}_{\substack{\text{which}\\ \text{depends}\\ \text{on}\\ \text{position}}} = \langle \underbrace{\partial_x f(x,y)}_{\substack{\text{the first}\\ \text{component}\\ \text{is partial}\\ \text{derivative}\\ \text{wrt first}\\ \text{variable}}}, \underbrace{\partial_y f(x,y)}_{\text{etc.}} \rangle$$

Higher dimension analog of derivative in 1d.

Example: $f(x,y) = x + y^2$

$\nabla f(x,y) = \langle 1, 2y \rangle$

Activity:

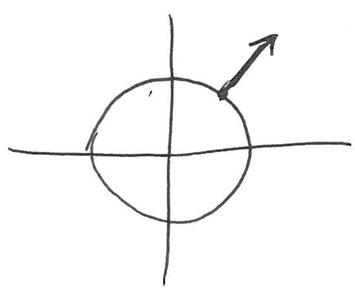
- Compute $\nabla f(x,y)$ for $f(x,y) = x^2+y^2$

- Draw level set containing $(1,1)$
- Plot $\vec{\nabla f}(1,1)$ w/ base of vector at $(1,1)$
-

$\nabla f = \langle 2x, 2y \rangle$

$\nabla f(1,1) = \langle 2, 2 \rangle$

At $(1,1)$ f is 2. The level set of value 2 is circle radius $\sqrt{2}$



b) $$\underbrace{f(\vec{x}+\vec{\Delta x})}_{\substack{\text{value of } f \\ \text{at nearby} \\ \text{point to } \hat{x}}} \approx f(\vec{x}) + \vec{\nabla f}(\vec{x}) \cdot \vec{\nabla x}$$

value of f at nearby point to $\hat{x}$ is value at $\hat{x}$ plus gradient · displacement

Just like 1d!

Let $\vec{x} = (x, y)$ $\quad \vec{\Delta x} = (\Delta x, \Delta y)$

Why?
$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + \partial_x f(x,y)\,\Delta x + \partial_y f(x,y)\,\Delta y$$
$$= f(x,y) + \langle \partial_x f, \partial_y f \rangle \cdot \langle \Delta x, \Delta y \rangle$$
$$= f(x,y) + \vec{\nabla f}(x,y) \cdot \langle \Delta x, \Delta y \rangle$$

Example: Approximate value of $f(x,y) = xy$ at $(0.8, 1.1)$ using value at $(1,1)$.

Want: $f(\langle 1,1 \rangle + \langle -0.2, 0.1 \rangle) \approx f(1,1) + \nabla f(1,1) \cdot \langle -0.2, 0.1 \rangle$

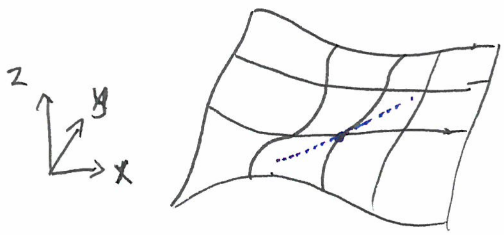
$f(1,1) = 1 \cdot 1 = 1$

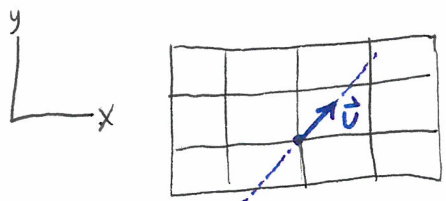
$\nabla f(x,y) = \langle y, x \rangle$

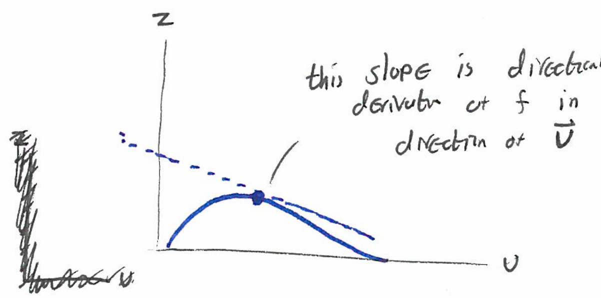
$\nabla f(1,1) = \langle 1,1 \rangle$

$$= 1 + \langle 1,1 \rangle \cdot \langle -0.2, 0.1 \rangle$$
$$\approx 1 - 0.2 + 0.1$$
$$= \boxed{0.9}$$

c) Directional derivative: Rate of change of $f(x,y)$ if (x,y) is changed in direction $\vec{U}$







$$D_U f = \lim_{\varepsilon \to 0} \frac{f(\vec{x} + \varepsilon \vec{U}) - f(\vec{x})}{\varepsilon}$$

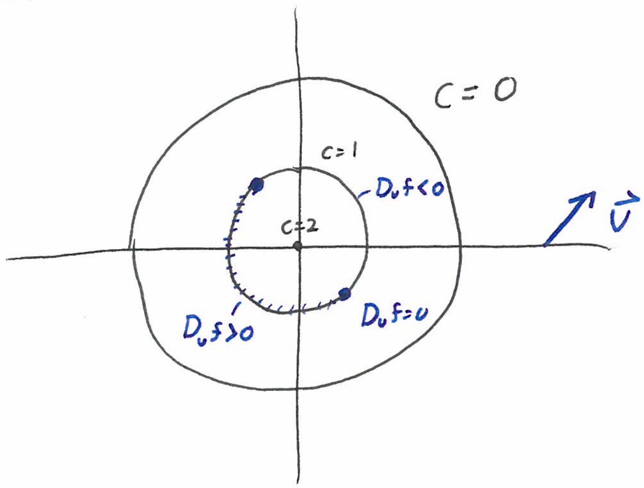
Scalar!

Activity: Along level curve for $c=1$, Draw where $D_{\vec{u}} f > 0$ when $\vec{u} = \frac{\langle 1,1 \rangle}{\sqrt{2}}$

$D_{\vec{u}} f = 0$

$D_{\vec{u}} f < 0$

Level sets of f.



d) $D_{\vec{v}} f = \vec{\nabla} f \cdot \vec{v}$ if $|\vec{v}| = 1$

$D_{\vec{v}} f = \vec{\nabla} f \cdot \dfrac{\vec{v}}{|\vec{v}|}$ if $|\vec{v}| \neq 1$

Why? $f(\vec{x} + \vec{\Delta x}) \approx f(\vec{x}) + \vec{\nabla} f \cdot \vec{\Delta x}$

So, $D_{\vec{v}} f = \lim_{\varepsilon \to 0} \dfrac{f(\vec{x} + \varepsilon \vec{v}) - f(\vec{x})}{\varepsilon}$

$= \lim_{\varepsilon \to 0} \dfrac{\vec{\nabla} f \cdot (\varepsilon \vec{v})}{\varepsilon}$

$= \vec{\nabla} f \cdot \vec{v}$

Example: Find $D_{\vec{v}} f(1,1)$ for $f(x,y) = x^2 + y^2$

$\vec{v} = \dfrac{\langle 1,2 \rangle}{\sqrt{5}}$

$$D_{\vec{v}} f(1,1) = \nabla f(1,1) \cdot \frac{\langle 1,2 \rangle}{\sqrt{5}}$$

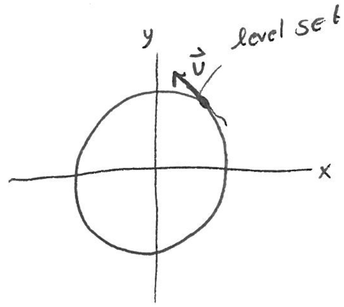
$$\nabla f(x,y) = \langle 2x, 2y \rangle$$

$$\nabla f(1,1) = \langle 2,2 \rangle$$

$$D_{\vec{v}} f(1,1) = \langle 2,2 \rangle \cdot \frac{\langle 1,2 \rangle}{\sqrt{5}} = \frac{6}{\sqrt{5}}.$$

e) $\nabla f(x,y) \perp$ level set of f at (x,y)

Why?



Moving tangent to level set, directional derivative is 0

(the level set all has constant value of f)

If $\vec{u}$ tangent to level set at (x,y), $|\vec{u}|=1$

$$D_{\vec{u}} f(x,y) = 0.$$

We know

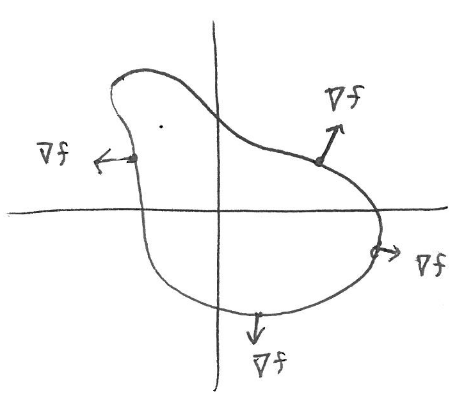
$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

So

$$\nabla f \cdot \vec{u} = 0.$$

Hence gradient $\perp$ level set.

Visually: Level sets



$\nabla f \perp$ level set.

Points toward higher values of f

Activity:

- Plot level set of value 0 for $f(x,y,z) = x^2+y^2-z$

- At $(1,1,2)$ plot $\nabla f(1,1,2)$ w/o computing it.

Relation of ∇f w/ steepest ascent:

$$D_{\vec{u}} f = \vec{\nabla f} \cdot \vec{u} \qquad (|\vec{u}| = 1)$$

$$= \underbrace{|\nabla f| \cos\theta}_{} \qquad \text{w/ } \theta \text{ angle between } \vec{u} \text{ and } \vec{\nabla f}$$

- largest positive value when $\theta = 0$ ie $\vec{u} \parallel \vec{\nabla f}$ $\quad \vec{u} = \frac{\vec{\nabla f}}{|\nabla f|}$
- largest neg value — $\theta = \pi$ $\quad \vec{u} \parallel -\vec{\nabla f}$ $\quad \vec{u} = -\frac{\vec{\nabla f}}{|\nabla f|}$

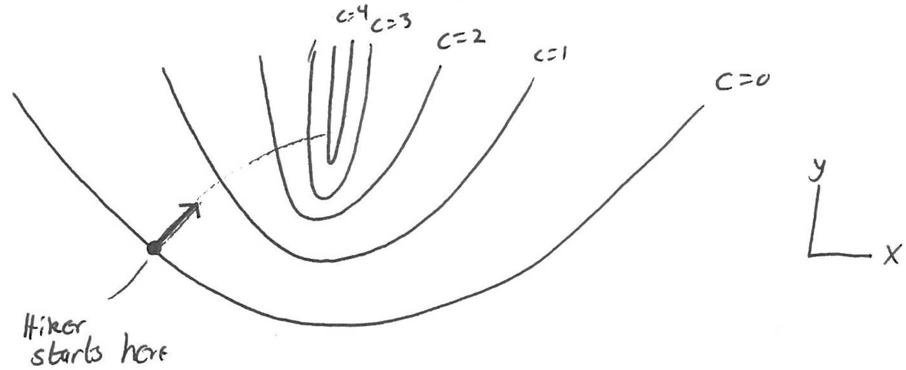
So, $|\nabla f|$ is direct. deriv in direction of steepest ascent

∇f is in direction of steepest ascent

$-\nabla f$ is in — — — descent

Activity

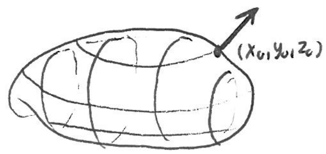
Level sets of a hill $z = f(x,y)$ are



a) Draw ∇f at initial point

b) Draw path of steepest ascent.

a) Normal vector to implicitly defined surface



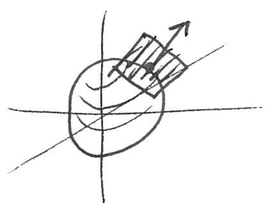
Surface given by $f(x,y,z)=C$

A Normal vector is $\nabla f(x_0,y_0,z_0)$

Why? Because $\nabla f \perp$ level set of f.
And surface is level set of f.

Example:

Find tangent plane to $x^2+y^2+z^2=3$ at $(1,1,1)$



Need $\vec{X_0}$ & $\vec{n}$

$\vec{X_0} = (1,1,1)$

$\vec{n}$ is normal to surface.

Surface is level set value 3 of $f(x,y,z) = x^2+y^2+z^2$

$\nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle$

$\nabla f(1,1,1) = \langle 2, 2, 2 \rangle$

Gradient gives vector $\perp$ surface

$\vec{n} = \langle 2,2,2 \rangle$

So $\vec{n} \cdot \vec{x} = \vec{n} \cdot X_0$

$\langle 2,2,2 \rangle \cdot \langle x,y,z \rangle = \langle 2,2,2 \rangle \cdot \langle 1,1,1 \rangle$

$\boxed{2x+2y+2z = 6}$

High dimensional Chain Rule & Total Derivative

Recall $\quad f(x+\Delta x,\ y+\Delta y) \approx f(x,y) + \partial_x f(x,y)\,\Delta x + \partial_y f(x,y)\,\Delta y$

Chain Rule: $\quad \partial_a\Big(f(x(a,b),\ y(a,b)) \Big) = \partial_x f\ \partial_a x + \partial_y f\ \partial_a y$

$$= \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial a} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial a}$$

how much f changes per change in x

how much x changes per change in a

Similarly for $\partial_b\, f(x(a,b), y(a,b))$.

Total derivative. Let $f(t,x,y,z)$ be temperature of ocean at time t & pos x,y,z.

If a submarine travels along curve $x(t), y(t), z(t)$, what is total derivative of temp wrt time?

$$\frac{d}{dt}\Big[f(t, x(t), y(t), z(t))\Big] = \partial_t f + \underbrace{\partial_x f\,\frac{dx}{dt} + \partial_y f\,\frac{dy}{dt} + \partial_z f\,\frac{dz}{dt}}_{\text{other vars } x,y,z \text{ are not held constant}}$$

Compare to partial deriv $\partial_t f$ – other vars held constant