Lecture 3 7 July 2014

Parameterized Curves

Lines

Circles

Complicated Motion

Velocity

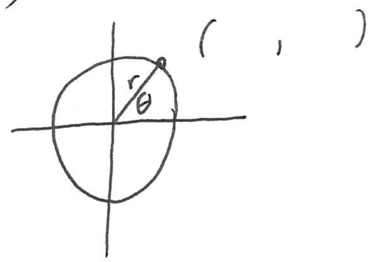
speed

Acceleration

Arc length

Warmup Activity

a)

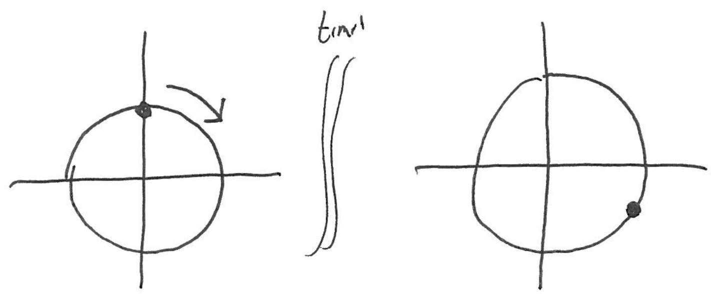


what are x, and y coordinates of point at angle θ radius r?

$x = r\cos\theta \quad y = r\sin\theta$

b) Suppose a merry go round of radius r rotates clock wise at ω rad/sec.

What is (x, y) coords at time t of point that is initially at $(0, r)$ at $t=0$.

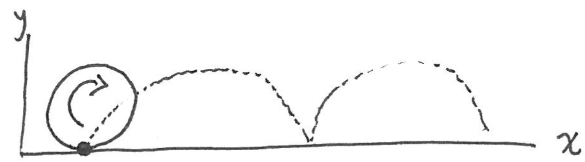


$$\theta(t) = \pi/2 - \omega t$$

$$\langle x(t), y(t) \rangle = r \langle \cos\theta(t), \sin\theta(t) \rangle$$
$$= \langle r\cos(\pi/2 - \omega t), r\sin(\tfrac{\pi}{2} - \omega t) \rangle$$

Need for parameterization:

Consider rolling disk



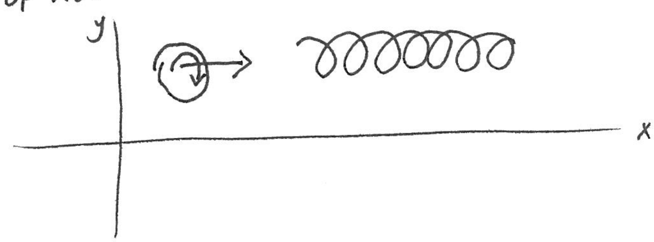
What is path traced out by point.

How to describe: $y(x)$? No (impossible w/ standard functions)

$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ yes

Top view

Frisbee



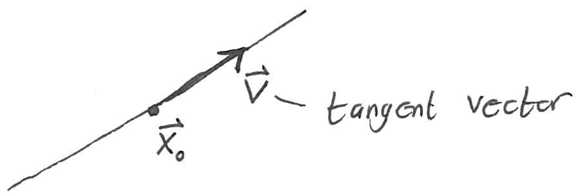
$y(x)$? No, multivalued

$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$? Yes

Need to describe translation & rotation

13 a) In n-d a line is given by

$$\vec{X}(t) = \vec{X}_0 + t\vec{V} \quad \text{for } -\infty < t < \infty$$



- A line is specified by a point and a direction
- Many ways to represent the same line

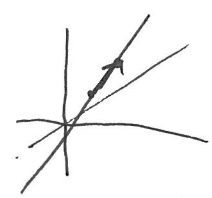
  Eg $\quad \vec{X}(t) = \vec{X}_0 + 2t\vec{V} \quad$ for $-\infty < t < \infty$

  is same line (just traced at twice the ~~speed~~ rate)

- Two points specify a line.

  Example: Find line going through $(1,0,1)$ & $(0,2,1)$

  Find tangent vector: $\langle 0,2,1\rangle - \langle 1,0,1\rangle = (-1,2,0)$

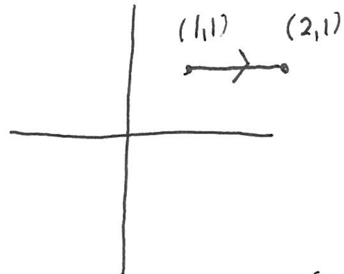


$$\vec{X}(t) = \langle 1,0,1\rangle + t\langle -1,2,0\rangle \quad \text{for } -\infty < t < \infty$$

example: line segment from $(1,0,1)$ to $(0,2,1)$

$$\vec{X}(t) = \langle 1,0,1\rangle + t\langle -1,2,0\rangle \quad \text{for } 0 \le t \le 1$$

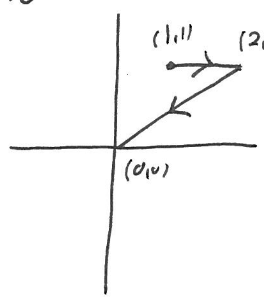
Activity:



Parameterize the line segment from (1,1) to (2,1)

$\vec{X}(t) = \underline{(1,1) + t\,(1,0)}$ for $\underline{0} \leq t \leq \underline{1}$

Parameterize the curve

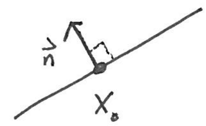


$$\vec{X}(t) = \begin{cases} (1,1) + t\,(1,0) & 0 \leq t \leq 1 \\ (2,1) + (t-1)(-2,1) & 1 \leq t \leq 2 \end{cases}$$

Specifying lines:

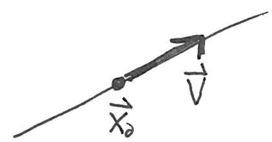
In 2d, line given by pt $\vec{X_0}$ and normal vector $\vec{n}$

$$\vec{n}\cdot\vec{X} = \vec{n}\cdot\vec{X_0}$$



In any dimension, line given by point $\vec{X_0}$ and tangent vector $\vec{V}$

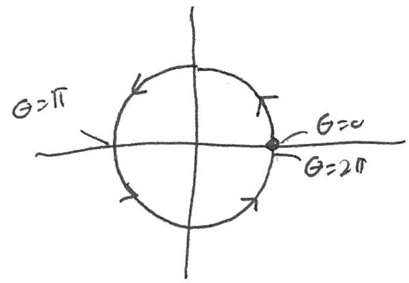
$$\vec{X}(t) = \vec{X_0} + t\,\vec{V}$$



Strengths and weaknesses

Parameterization of circle

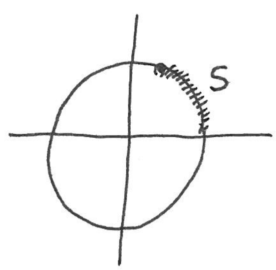
$\vec{X}(\theta) = \langle r\cos\theta, r\sin\theta \rangle \qquad 0 \le \theta < 2\pi$ traces out a circle counterclockwise



$\vec{X}(t) = \langle r\cos wt, r\sin wt \rangle \qquad 0 \le t < \frac{2\pi}{w}$ traces same circle at rate w rad/sec

Example: parameterize circle by arc length, starting at $(r,0)$

Find $\vec{X}(s)$



$\theta = s/r$

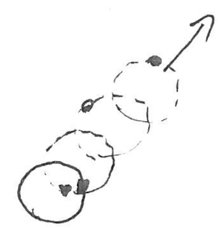
$\vec{X}(s) = \langle r\cos\frac{s}{r}, r\sin\frac{s}{r} \rangle$ for $0 \le s < 2\pi r$

Parameterizations of complex motions

To parameterize a curve consisting of multiple parts (translation + rotation + ...):

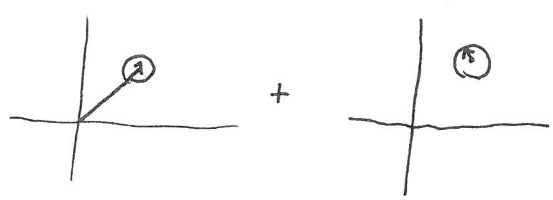
(1) • Describe ~~total~~ desired point as vector sum of each component
(2) • Identify convenient parameter. when in doubt use time
(3) • Identify each components dependence on parameters

Example: Frisbee of radius r initially centered at $(0,0)$ Translates in direction $(1,1)$ with speed V. Rotates at w rad/sec. c.c.w. Point originally at $(r,0)$ is painted. what curve is traced out?



(1) Sum of translation + rotation

$$\vec{X} = \vec{X}_{center} + \vec{X}_{rot}$$



(2) Given rates w.r.t. time, so use t

(3) $\vec{X}_{center}(t) = (1,1)\, t$? No. Didnt use V.

what is vector in dir of $(1,1)$ w/ length vt ?

$$\vec{X}_{center}(t) = \underbrace{\frac{(1,1)}{|(1,1)|}}_{\text{dir } (1,1)} vt = \cancel{\frac{vt,\ vt}{\sqrt{2}}} \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) vt$$

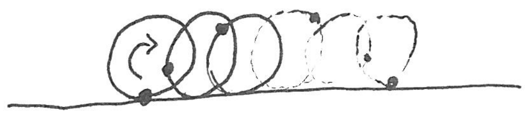
$\vec{X}_{robot}(t) = (r\cos wt,\ r\sin wt)$ Note, checks at at $t=0$

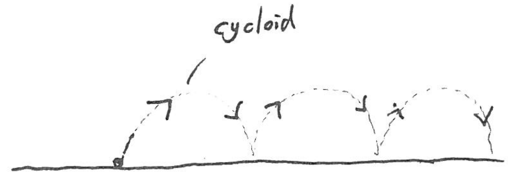
Combining

$$\boxed{\vec{X}(t) = \left(\tfrac{1}{\sqrt{2}}vt + r\cos wt,\ \tfrac{1}{\sqrt{2}}vt + r\sin wt\right) \quad -\infty < t < \infty}$$

Example: Cycloid.

Cylinder radius r rolls along flat ground at angular speed ω. Initially point touching ground is painted. As it rolls, what path is traced out?

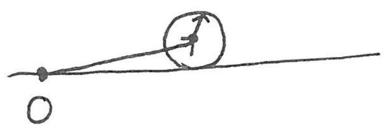




1) Break motion into separate parts:

center of cylinder translates + cylinder rotates

$$\vec{X} = \vec{X}_{c.enter} + \vec{X}_{rotation}$$



2) Identify each's time dependence

After time t, angle rotated is ωt

distance traveled is $r\omega t$

$$\vec{X}_{com}(t) = (r\omega t, r)$$

Angle relative to center is $\theta(t) = \underbrace{-\frac{\pi}{2}}\ \underbrace{-\ \omega t}$

because initially point is down from center

why minus? Because positive angles are C.C.W

$$\vec{X}_{rot}(t) = (r\cos\theta(t), r\sin\theta(t))$$

$$= (r\cos(-\tfrac{\pi}{2}-\omega t), r\sin(-\tfrac{\pi}{2}-\omega t))$$

$$\boxed{\vec{X} = (r\omega t + r\cos(-\tfrac{\pi}{2}-\omega t), r + r\sin(-\tfrac{\pi}{2}-\omega t)) \qquad -\infty < t < \infty}$$

Velocity, Speed, Acceleration

If $\vec{X}(t)$ is parameterization of position wrt time,

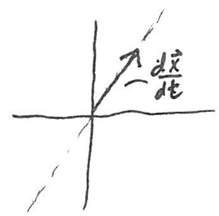
$\vec{V}(t) = \frac{d}{dt}\vec{X}(t)$ is velocity (vector) at time t

$|\vec{V}(t)|$ is speed (scalar) at time t

$\vec{a}(t) = \frac{d^2\vec{X}(t)}{dt^2} = \frac{d\vec{V}(t)}{dt}$ is acceleration

Note: $\vec{V}(t)$ provides tangent vector to curve

Example: Find velocity, speed, acceleration of $\vec{X}(t) = (2t, 3t)$



$\frac{d\vec{x}}{dt} = (2,3)$ — velocity

$\left|\frac{d\vec{x}}{dt}\right| = \sqrt{2^2+3^2} = \sqrt{13}$ — speed

$\frac{d^2\vec{x}}{dt^2} = (0,0)$ — acceleration vector

Example: Same for circular motion (radius r, w rad/sec)

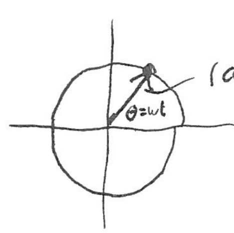
$\vec{X}(t) = (r\cos wt, r\sin wt)$

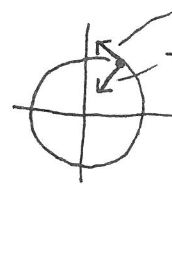
$\frac{d\vec{x}}{dt}(t) = (-rw\sin wt, rw\cos wt)$

$\left|\frac{d\vec{x}}{dt}\right|(t) = \sqrt{r^2w^2(\sin^2 wt + \cos^2 wt)} = rw$ — units work out

$\frac{d^2\vec{x}}{dt^2}(t) = (-r^2w^2\cos wt, -r^2w^2\sin wt)$

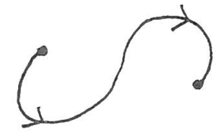
$= -r^2w^2(\cos wt, \sin wt)$ — allways points toward origin





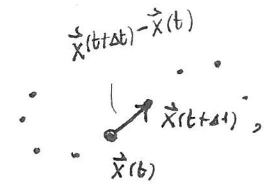
Why is velocity tangent to curve?

Let $\vec{X}(t)$ be a curve



$$\vec{V}(t) = \frac{d\vec{X}(t)}{dt}$$

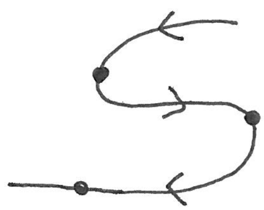
$$= \lim_{\Delta t \to 0} \frac{\vec{X}(t+\Delta t) - \vec{X}(t)}{\Delta t}$$



$$\approx \frac{\vec{X}(t+\Delta t) - \vec{X}(t)}{\Delta t}$$ which is in direction along curve

Activity:

Consider following parametric curve



Draw velocity & acceleration vectors at designated points

Arclength

To find length of a parameterized curve, integrate speed.

Arc length of $\vec{X}(t)$ from $t=a$ to b is

$$S = \int_a^b \left| \frac{d\vec{X}(t)}{dt} \right| dt$$ ~ very easy to remember

Example: Perimeter of circle radius r.

Parameterize $\vec{X}(t) = (r\cos\omega t, r\sin\omega t)$ $\quad 0 \leq t \leq 2\pi/\omega$

$$\frac{d\vec{X}}{dt} = +r\omega(-\sin\omega t, \cos\omega t) \qquad \left|\frac{d\vec{X}}{dt}\right| = r\omega$$

$$S = \int_0^{2\pi/\omega} r\omega\, dt = r\omega \frac{2\pi}{\omega} = 2\pi r.$$

Note: arc length formula works for any parameter (not just for time)

~~Example~~

$$S = \int_a^b \left| \frac{d\vec{X}}{d\theta}(\theta) \right| d\theta$$

~~Example:~~

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Problem Set 2

Due: **14 June 2014** in class.

1. (10 points)

   (a) Let $\mathbf{a}(t)$ and $\mathbf{b}(t)$ give curves in 3d. By writing out the components explicitly, show that

   $$\frac{d}{dt}\Big(\mathbf{a}(t)\cdot\mathbf{b}(t)\Big) = \frac{d\mathbf{a}}{dt}\cdot\mathbf{b} + \mathbf{a}\cdot\frac{d\mathbf{b}}{dt}$$

   (b) Show that if the speed of an object is constant, then its acceleration is always perpendicular to its velocity.
   Hint: If the object has velocity $\mathbf{v}(t)$, study $\frac{d}{dt}|\mathbf{v}(t)|^2$.

2. (20 points) Parameterizations

   (a) Parameterize the square traversed from $(0,0)$ to $(0,1)$ to $(1,1)$ to $(1,0)$ and back to $(0,0)$. Your answer should be expressed as a piecewise linear function $\mathbf{X}(t)$.

   (b) A gear of radius R is centered at the origin and fixed so that it can not move or rotate. Another gear of radius r initially touches the fixed gear at $(-R,0)$ and rotates clockwise around the fixed gear. Sketch and find the trajectory of the point originally touching the fixed gear.

3. (15 points) Let $\mathbf{X}(t) = (e^{-t}\cos t, e^{-t}\sin t)$ for $0 \le t < \infty$.

   (a) Plot this curve.

   (b) Compute the velocity vector and speed as functions of t.

   (c) Compute the length of the curve.

   (d) How many times does the curve circle the origin?

   (e) Does the contrast of (c) and (d) surprise you?

   (f) Extra credit (5 points): Find a curve of the form $(r(t)\cos t, r(t)\sin t)$ that spirals into the origin and has infinite arc length. Justify your answer with a calculation.