Lecture 2 2 july 2014

Lines

Planes

Cross Product

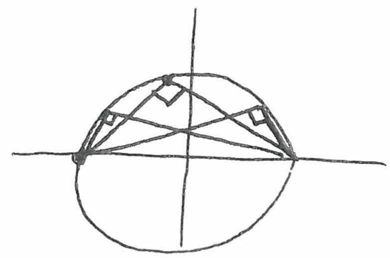
Warm up

In 2d, find a vector $\perp$ to $\hat{\imath}+\hat{\jmath}$.

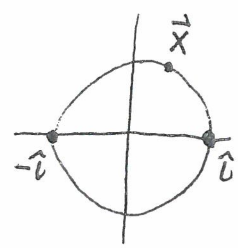
In 3d, find two vectors $\perp$ to $\hat{\imath}+\hat{\jmath}$ and perpendicular to each other.

10) Geometric Proofs

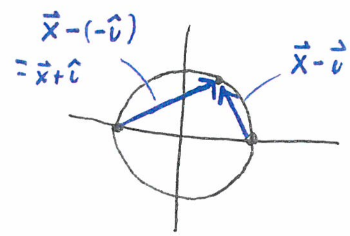
Example: Show that lines connecting any point on the unit ~~semi~~circle to $(1,0)$ and $(-1,0)$ are perpendiculr



Step 1: Label important points



Step 2: Identify important vectors



Step 3: Interpret givens and goal in terms of vectors

Given: $|\vec{x}|=1$

Goal: $\vec{x}+\hat{\imath} \perp \vec{x}-\hat{\imath}$

Step 4: Connect givens and goal by algebra

Want to show $(\vec{x}+\hat{\imath})\cdot(\vec{x}-\hat{\imath})=0$.

Computing,

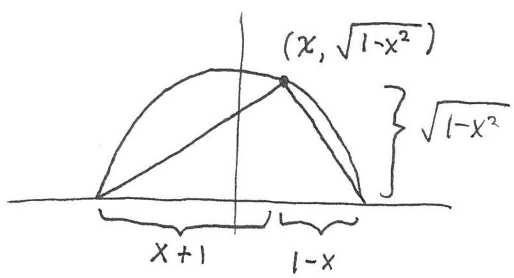
$$(\vec{x}+\hat{\imath})\cdot(\vec{x}-\hat{\imath}) = \vec{x}\cdot\vec{x} - \cancel{\vec{x}\cdot\hat{\imath}} + \cancel{\hat{\imath}\cdot\vec{x}} - \hat{\imath}\cdot\hat{\imath}$$
$$= |\vec{x}|^2 - 1$$
$$= 1 - 1 = 0.$$

Did Vectors make this problem easy?

Yes, nothing but interpreting givens & goal in terms of vectors.

Other approaches involve a clever decomposition of the triangle or involve slopes.

Approach 2:



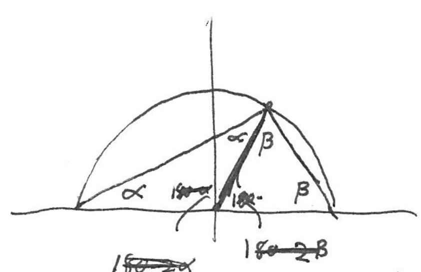
Slopes: $\dfrac{\sqrt{1-x^2}}{1+x}$ & $\dfrac{-\sqrt{1-x^2}}{1-x}$

$\sqrt{\dfrac{1-x}{1+x}}$ & $-\sqrt{\dfrac{1+x}{1-x}}$

slopes m & $-\frac{1}{m}$ are perpendicular

annoying algebra & had to solve eqn for circle

Approach 3:



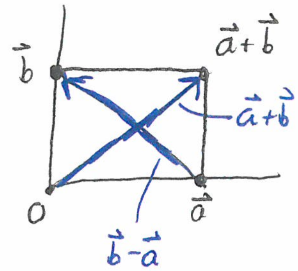
$2\alpha + 2\beta = \pi$

So $\boxed{\alpha + \beta = \frac{\pi}{2}}$

requires clever way to view triangle

Example: Show that a rectangle is a square if its diagonals are perpendicular

Label points & vectors



Given: $\vec{a} \perp \vec{b}$ and $\vec{a}+\vec{b} \perp \vec{b}-\vec{a}$

Goal: $|\vec{a}| = |\vec{b}|$

Algebra: $(\vec{a}+\vec{b})\cdot(\vec{b}-\vec{a}) = 0$

$\Rightarrow \cancel{\vec{a}\cdot\vec{b}} - \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - \cancel{\vec{b}\cdot\vec{a}} = 0$

$\Rightarrow \quad \vec{b}\cdot\vec{b} = \vec{a}\cdot\vec{a}$

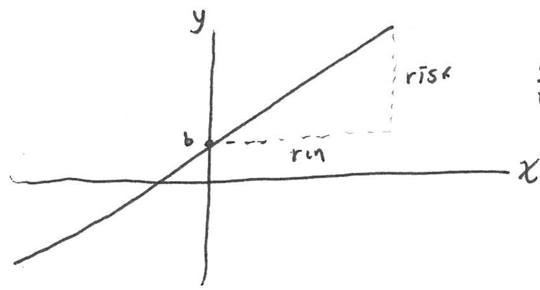
$\Rightarrow \quad |\vec{b}|^2 = |\vec{a}|^2$

$\Rightarrow \quad |\vec{b}| = |\vec{a}|$ ∎

11) Lines in 2d

A line of slope m and y-intercept b is given by

$$y = mx + b$$

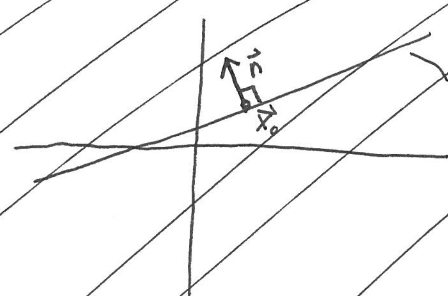


$\frac{\text{rise}}{\text{run}} = m$

This is cumbersome & incomplete. What if line is vertical?

Slope distinguishes x from y. Want a way to treat x and y <u>symmetrically</u>

In 2d, a line is specified with a point $\vec{X}_0$ and normal vector $\vec{n}$.



$\vec{n} \cdot \vec{X} = \vec{n} \cdot \vec{X}_0$

Why? $\vec{n} \cdot (\vec{X} - \vec{X}_0) = 0$ means $\vec{n} \perp \vec{X} - \vec{X}_0$. Only one direction (& its opposite) is $\perp$.

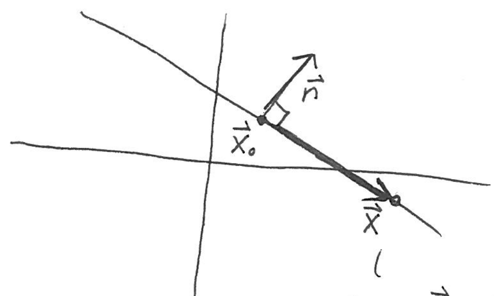
If $\vec{X} - \vec{X}_0 \perp \vec{n}$, so is any scaler multiple. Hence, a line.

11)

In 2d, any line satisfies $\vec{n}\cdot\vec{x} = b$

normal vector

If line contains $\vec{x}_0$, then $\vec{n}\cdot\vec{x}_0 = b$, and line is given by $\vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{x}_0$.

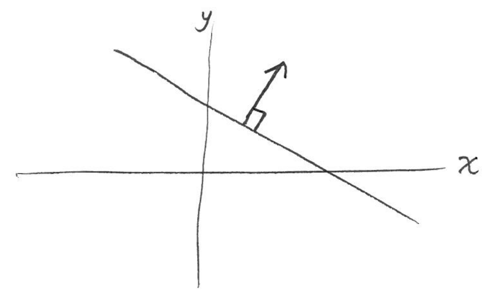


any $\vec{x}$ along this line is such that

$$\vec{x}-\vec{x}_0 \perp \vec{n} \iff \vec{n}\cdot(\vec{x}-\vec{x}_0) = 0$$
$$\iff \vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{x}_0.$$

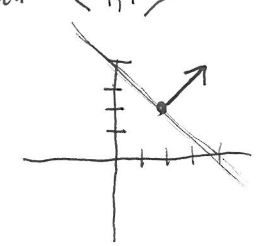
In this form, Easy to read off normal vector of a line

Example: In 2d, $2x+3y=5$ defines a line w/ normal vector $\langle 2,3\rangle$



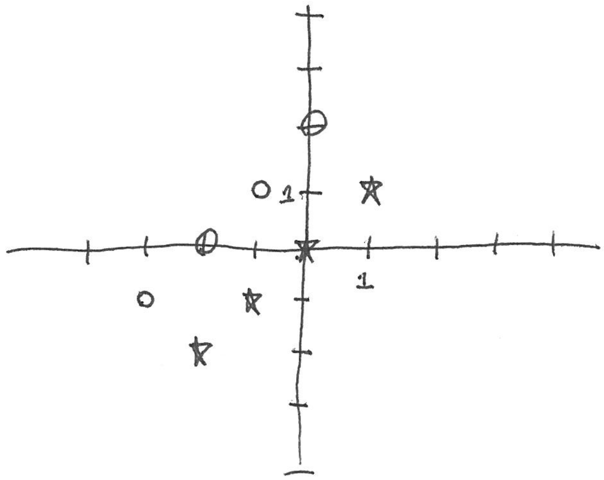
$$\underbrace{\langle 2,3\rangle}_{\vec{n}} \cdot \underbrace{\langle x,y\rangle}_{\vec{x}} = 5$$

Example: line going through (2,2) with normal vector $\langle 1,1\rangle$



$$\langle 1,1\rangle\cdot\langle x,y\rangle = \langle 1,1\rangle\cdot\langle 2,2\rangle$$
$$x+y = 4$$

Activity:



Find a line of form $\vec{n} \cdot \vec{x} = b$
so that O's and ☆'s are on opposite
sides,

Identify $\vec{n}$ and $\vec{X_0}$

Discussion:

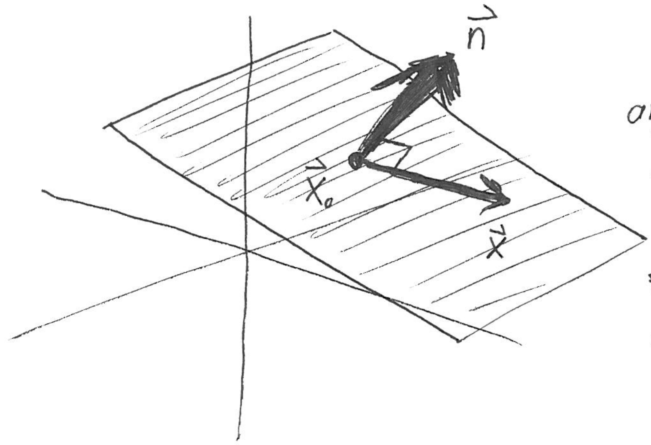
Why is vector view better than slope view?

In 3d, any plane satisfies $\vec{n} \cdot \vec{x} = b$

normal vector

If plane contains $\vec{x}_0$, then $\vec{n} \cdot \vec{x}_0 = b$, and plane given by

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$$



any $\vec{x}$ on plane is such that

$\vec{x} - \vec{x}_0 \perp \vec{n}$

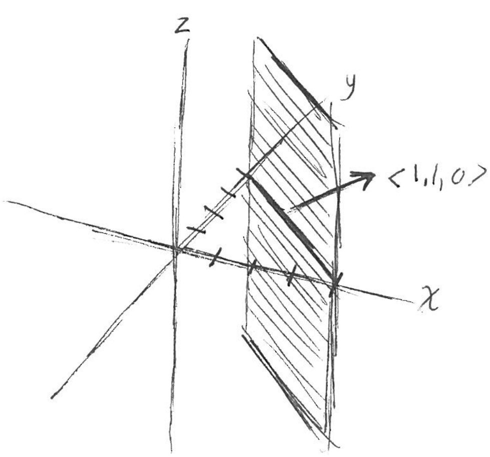
$\Rightarrow \vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$

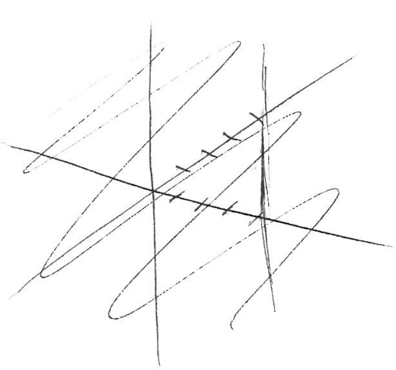
$\Rightarrow \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$

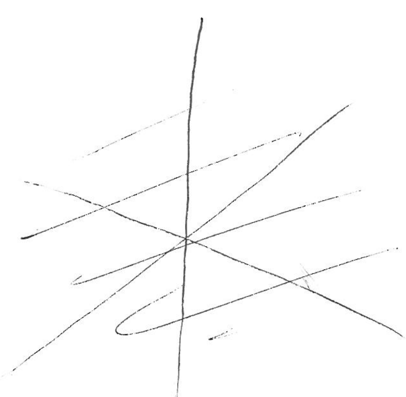
Example: plane with normal vector $\langle 1,1,0 \rangle$ going through $(2,2,0)$ is

$$\langle 1,1,0 \rangle \cdot \langle x,y,z \rangle = \langle 1,1,0 \rangle \cdot \langle 2,2,0 \rangle$$

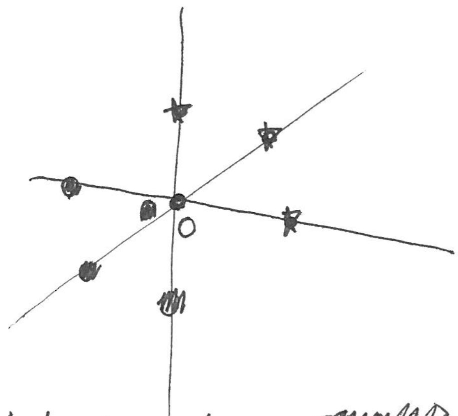
$$x + y = 4$$







Activity:



Find a plane that separates $\langle 0,0,0\rangle$ from $\langle 1,0,0\rangle$, $\langle 0,1,0\rangle$, $\langle 0,0,1\rangle$.

Hint: Think of normal vector and a point in plane.

12) Matrices

A matrix is a rectangular collection of numbers

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

2×2 $\qquad$ 3×3

May be nonsquare

Determinant of a 2×2 matrix

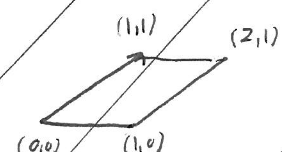
$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

diagonals minus off-diagonals

Example $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 2 = 0.$

Geometrically, $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the area of parallelogram spanned by $\langle a,b\rangle$ and $\langle c,d\rangle$. (up to a minus sign)

Example: area of



$= \det\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$= 1\cdot 0 - 1\cdot 1 = -1.$

Area is $\boxed{1}$

Discussion on how to solve without vectors.

How will you remember if area of

is $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ or $\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}$?

Cross Product

If $\vec{X} = \langle x_1, x_2, x_3 \rangle$ and $\vec{y} = \langle y_1, y_2, y_3 \rangle$

then $\vec{X} \times \vec{y} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$

first vector goes here

second vector goes here

$$= \hat{\imath} \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \hat{k} \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

note minus sign

Ex: Compute $\langle 1, 2, 0 \rangle \times \langle -1, 0, 1 \rangle$.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \hat{\imath} \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix}$$

$$= \hat{\imath}\, 2 - \hat{\jmath}\, 1 + \hat{k}\,(2)$$

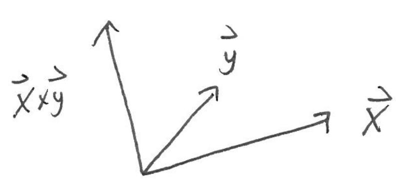
$$= \langle 2, -1, 2 \rangle$$

Cross product of $\vec{x}$ & $\vec{y}$ is perpendicular to $\vec{x}$ & $\vec{y}$

Example: $\langle 1, 2, 0 \rangle \times \langle -1, 0, 1 \rangle = \langle 2, -1, 2 \rangle$

Verify: $\langle 1, 2, 0 \rangle \cdot \langle 2, -1, 2 \rangle = 2 - 2 + 0 = 0$ ✓

$\langle -1, 0, 1 \rangle \cdot \langle 2, -1, 2 \rangle = -2 + 0 + 2 = 0$ ✓



Activity:

$$\langle 1,1,1\rangle \times \langle 2,2,2\rangle$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= 0$$

Why must it be 0?

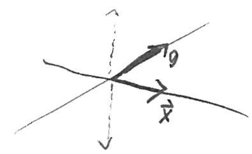
$$\langle 1,0,2\rangle \times \langle 0,1,2\rangle$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{vmatrix} = \hat{i}\begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= -2\hat{i} - 2\hat{j} + \hat{k}$$

Right hand rule

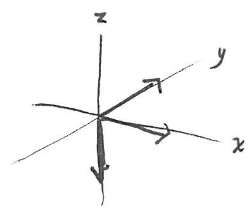
$\vec{X} \times \vec{y}$ is perpendicular to $\vec{x}$ & $\vec{y}$, but there are two such directions



which one is it?

The one given by right hand rule.

Rule: line right hand fingers along $\vec{x}$
curl them directly toward $\vec{y}$,
Thumb points along $\vec{X} \times \vec{y}$



Example: $\hat{j} \times \hat{i} = -\hat{k}$

$\hat{i} \times \hat{k} = -\hat{j}$

Activity:

Apply right hand rule to determine

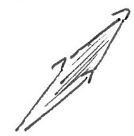
Is $(\hat{i}+\hat{j})\times\hat{k} = \hat{i}-\hat{j}$ or $\hat{j}-\hat{i}$?

Is $\hat{j}\times\hat{k} = \hat{i}$ or $-\hat{i}$?

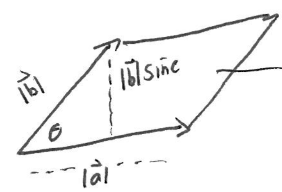
Length of cross product

$|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$, where θ is angle between $\vec{a}$ & $\vec{b}$

$= \text{area of parallelogram in 3D spanned by } \vec{a} \text{ \& } \vec{b}$



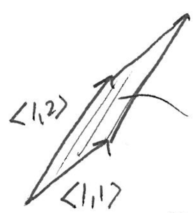
$\vec{a}\times\vec{b} = \vec{0}$ if. $\vec{a}$ & $\vec{b}$ parallel ($\theta = 0$ or π)



$A = |\vec{a}|\,|\vec{b}|\sin\theta$.

~~Example:~~

Example:



area $= |\langle 1,1,0\rangle \times \langle 1,2,0\rangle|$

$$\langle 1,1,0\rangle \times \langle 1,2,0\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= \hat{k}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = \hat{k}$$ – has length 1

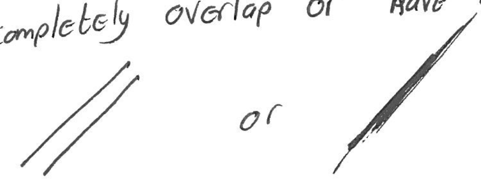
So area of parallelogram is $\boxed{1}$

Discussion of how to do without vectors.

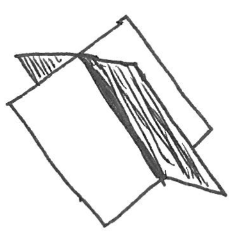
Intersections of Lines and Planes

In 2D, Two lines typically intersect in a point

May completely overlap or have no overlap if parallel



In 3d, two planes typically intersect in a line



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