Lecture 1 - Vectors, dot products, geometric proofs - 6/30/2014 — Interphase 2014 Calc 3

1. Vectors

   a. A vector is a quantity with a direction and magnitude. A scalar is just magnitude (which may be negative).

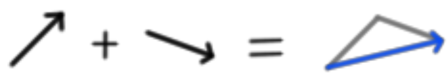
   b. A vector is a collection of multiple numbers. A scalar is a single number.

   c. Vector notation: $\mathbf{x} = \vec{x} = (x_1, x_2, x_3) = \langle x_1, x_2, x_3 \rangle = x_1\hat{\mathbf{i}} + x_2\hat{\mathbf{j}} + x_3\hat{\mathbf{k}}$

2. Vector addition

   a. Vectors add componentwise: $\langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle = \langle x_1 + y_1, x_2 + y_2 \rangle$.

   b. The sum of two vectors is the vector obtained by lining up the tail of one vector to the head of the other:

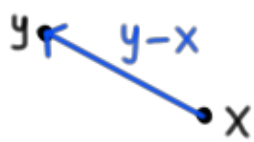


   c. When an object has a velocity relative to a moving medium, it's net velocity is the sum of it's relative velocity and the medium's velocity.

3. Vector subtraction

   a. Vectors subtract componentwise: $\langle y_1, y_2 \rangle - \langle x_1, x_2 \rangle = \langle y_1 - x_1, y_2 - x_2 \rangle$.

   b. The vector from $\mathbf{x}$ to $\mathbf{y}$ is given by $\mathbf{y} - \mathbf{x}$.



   c. Subtracting a vector is the same as adding the negative of the vector:

$$\mathbf{y} - \mathbf{x} = \mathbf{y} + (-\mathbf{x}).$$

4. Scalar-vector multiplication

   a. Scalar-vector multiplication applies componentwise: $c\langle x_1, x_2, x_3 \rangle = \langle cx_1, cx_2, cx_3 \rangle$.

   b. $c\mathbf{x}$ is the vector obtained by multiplying the length of $\mathbf{x}$ by c:

5. Length of vectors

   a. The length of $\langle a, b, c\rangle$ is $|\langle a, b, c\rangle| = \sqrt{a^2 + b^2 + c^2}$.

   b. A unit vector is one with length 1.

   c. Scalar multiplication scales vector length: $|c\mathbf{x}| = |c||\mathbf{x}|$.

6. Direction of vectors

   a. The direction of a vector is the unit vector the points in the same 'direction':

$$\text{dir}\,\mathbf{u} = \frac{\mathbf{u}}{|\mathbf{u}|}\,.$$

   b. Any vector is its length times its direction: $\mathbf{u} = |\mathbf{u}|\,\text{dir}\,\mathbf{u}$.

7. Angle between vectors

   a. The angle between two vectors is the angle swept by the arc that directly connects them, provided that the vectors share the same base.

   b. The angle between vectors is always between 0 and π, inclusive. It is 0 if the vectors are in the same direction and π if the vectors are in opposite directions.

   c. Vectors are perpendicular if the angle between them is $\pi/2$.

8. Dot product

   a. Dot product definition: $\langle x_1, x_2, x_3\rangle \cdot \langle y_1, y_2, y_3\rangle = x_1y_1 + x_2y_2 + x_3y_3$.

   b. $\mathbf{x} \cdot \mathbf{x} = |\mathbf{x}|^2$.

   c. Dot product properties:

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= \mathbf{y} \cdot \mathbf{x} \\ (\mathbf{u} + \mathbf{v}) \cdot \mathbf{x} &= \mathbf{u} \cdot \mathbf{x} + \mathbf{v} \cdot \mathbf{x} \\ (c\mathbf{u}) \cdot \mathbf{v} &= \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v}) \end{aligned}$$

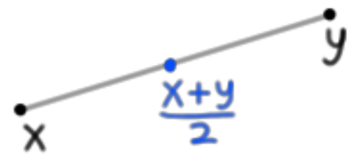
   d. Dot product angle formula: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos\theta$, where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$.

   e. Vectors are perpendicular if and only if their dot product is zero.

   f. The dot product is a measure of how far one vector is along the other. It is often used as a measure of similarity of vectors.

9. Other

a. The midpoint between **x** and **y** is $\frac{\mathbf{x}+\mathbf{y}}{2}$:



b. You can not add a scalar and a vector. You may only add two scalars or two vectors.

c. Standard arithmetic functions take scalars as arguments. For example, the log of scalar makes sense, and the log of a vector does not.

10. Geometric proofs

a. Begin a geometric proof by labeling important points with as few variables as possible.

b. To complete a geometric proof, interpret the assumptions and result in terms of vectors. Then try to connect them. Avoid componentwise thinking, if possible.