

Lecture 12

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Fundamental Theorem of Line Integrals
Green's Theorem

30 July 2014

Warmup

Find $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = \langle 1, 2 \rangle$
and C the line segment
connecting $(0,0)$ to $(3,1)$.

$$\vec{r}(t) = (0,0) + t(3,1) \text{ for } 0 \leq t \leq 1$$

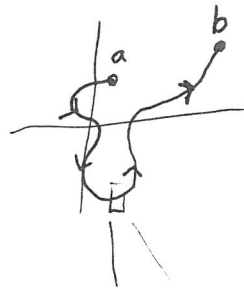
$$\frac{d\vec{r}}{dt}(t) = \langle 3, 1 \rangle$$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_C \vec{F} \cdot \frac{d\vec{r}}{dt} dt \\ &= \int_0^1 \langle 1, 2 \rangle \cdot \langle 3, 1 \rangle dt = \int_0^1 5 dt = 5. \end{aligned}$$

Fundamental theorem of Line Integrals

$$\int_C \nabla \psi \cdot d\vec{r} = \psi(b) - \psi(a)$$

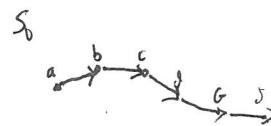
If C is any path connecting a & b



Justification:

$$\int_C \underbrace{\nabla \psi \cdot \vec{T}}_{\text{directional derivative of } \psi \text{ in direction of curve}} ds$$

$$\nabla \psi \cdot \vec{T} ds = \Delta \psi$$



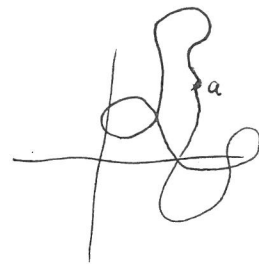
$$\psi(b) - \psi(a) + \psi(c) - \psi(b) + \dots + \psi(f) - \psi(e) = \psi(f) - \psi(a)$$

Compare to Fundamental Thm of Calc $\int_a^b f'(t) dt = f(b) - f(a)$

Implication:

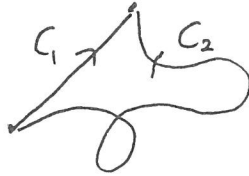
Line Integral of conservative vector fields over closed curves is 0.

$$\oint_C \nabla \psi \cdot d\vec{r} = \psi(a) - \psi(a) = 0$$



line integral over closed curve with positive orientation

Line integrals of conservative vector fields
are path independent:



Activity

Find $\int_C \langle 1, 2 \rangle \cdot d\vec{r}$ for C ^{the} line segment connecting $(0, 0)$ to $(3, 1)$.

$$\vec{F} = \langle 1, 1 \rangle.$$

Find potential function

$$\begin{aligned} \partial_x \phi &= 1 & \Rightarrow & \phi = x + 2y \\ \partial_y \phi &= 2 & \Rightarrow & \end{aligned}$$

$$\int_C \nabla(\phi) \cdot d\vec{r} = \phi(3, 1) - \phi(0, 0) = 5 - 0 = 5.$$

Example: Find $\int_C \vec{F} \cdot d\vec{r}$

$$\text{for } \vec{F} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}}$$

for C is ^{circular arc} ~~line~~ from $(R, 0)$ to $(0, R)$.

Method I: Fundamental Theorem

$$\vec{F} = \nabla \sqrt{x^2 + y^2} \quad \text{so } \phi(x, y) = \sqrt{x^2 + y^2}$$

$$\int_C \vec{F} \cdot d\vec{r} = \phi(0, R) - \phi(R, 0) = R - R = 0.$$

Method II: Directly $0 \leq \theta \leq \pi/2$

$$\vec{r}(\theta) = \langle R \cos \theta, R \sin \theta \rangle \quad \frac{d\vec{r}}{d\theta} = \langle -R \sin \theta, R \cos \theta \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \frac{\langle R \cos \theta, R \sin \theta \rangle}{R} \cdot \langle -R \sin \theta, R \cos \theta \rangle d\theta$$

$$= R \int_0^{\pi/2} \underbrace{\langle \cos \theta, \sin \theta \rangle \cdot \langle -\sin \theta, \cos \theta \rangle}_0 d\theta$$

$$= 0.$$

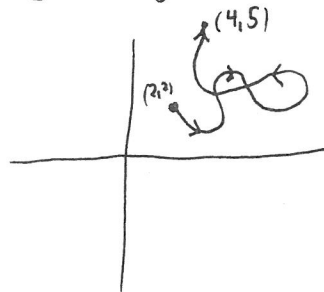
Example:

Find $\int_C \vec{F} \cdot d\vec{s}$ for $\vec{F} = x\vec{i} + y\vec{j}$

& C is any curve from $(2,2)$ to $(4,5)$

Notice $\vec{F} = \nabla(\frac{1}{2}x^2 + \frac{1}{2}y^2)$

$$\phi = \frac{1}{2}x^2 + \frac{1}{2}y^2$$



$$\begin{aligned} \int_C \vec{F} \cdot d\vec{s} &= \phi(4,5) - \phi(2,2) \\ &= \frac{1}{2}(4)^2 + \frac{1}{2}(5)^2 - (\frac{1}{2}(2)^2 + \frac{1}{2}(2)^2) \end{aligned}$$

$$= \boxed{\frac{33}{2}}$$

Closed, Simple, Positively Oriented Curves

A curve C is parameterized as $\vec{r}(t)$ for $a \leq t \leq b$

- C is closed if $\vec{r}(b) = \vec{r}(a)$ "starts where it ends"

- C is simple if it has no self intersections



Simple



Not simple

- positively oriented if interior of shape is on left.
(counterclockwise)



Positively oriented



No

- piecewise smooth:



Yes



No

Green's Theorem

Let C be positively oriented, piecewise smooth
simple closed curve that bounds region R

$$\oint_C P dx + Q dy = \iint_R (\partial_x Q - \partial_y P) dA$$

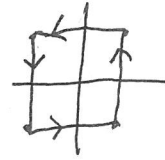


- Converts between line integrals
& area integrals

Example 0

$$\oint_C (x+y^2)dx + (y+x^2)dy$$

v/ $C =$ square with vertices $(\pm 1, \pm 1)$



$$P = x+y^2$$

$$Q = y+x^2$$

By Green's Thm

$$\oint_C P dx + Q dy = \iint_R (\partial_x Q - \partial_y P) dA$$

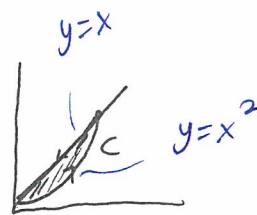
$$= \iint_R (2x - 2y) dA$$

$$= 2 \iint_R x dA - 2 \iint_R y dA$$

$$= 0 - 0 = 0$$

Example

$$\oint_C (x^2 - y^2) dx + (xy) dy$$



$$\oint_C P dx + Q dy = \iint_R (\partial_x Q - \partial_y P) dA$$

$$= \iint_R (y - 2y) dA$$

$$= - \iint_R y dA = - \int_0^1 \int_{x^2}^x y dy dx$$

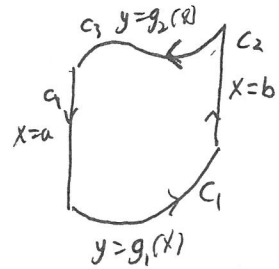
$$= - \int_0^1 \left[\frac{1}{2} x^2 - \frac{x^4}{2} \right] dx$$

$$= -\frac{1}{2} \left[\frac{1}{3} x^3 - \frac{1}{5} x^5 \right]_0^1$$

$$= -\frac{1}{2} \left[\frac{1}{3} - \frac{1}{5} \right] = -\frac{1}{15}$$

Proof of Green's Theorem in Simple Region

Suppose R is vertically simple



$$\oint_C P dx = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} P dx$$

On C_2 & C_4 $dx=0$, so $\int_{C_2} P dx = 0$
 $\int_{C_4} P dx = 0$

$$\begin{aligned} \oint_C P dx &= \int_{C_1} P dx + \int_{C_3} P dx \\ &= \int_a^b P(x, g_1(x)) dx - \int_a^b P(x, g_2(x)) dx \\ &= - \int_a^b (P(x, g_2(x)) - P(x, g_1(x))) dx \\ &= - \int_a^b \int_{g_1(x)}^{g_2(x)} \partial_y P(x, y) dy dx = - \iint_R \partial_y P dA \end{aligned}$$

Similarly $\int_C Q dy = \iint_R \partial_x Q dA$ for horizontally simple regions

Proof in complicated domains: Break into simple subregions

