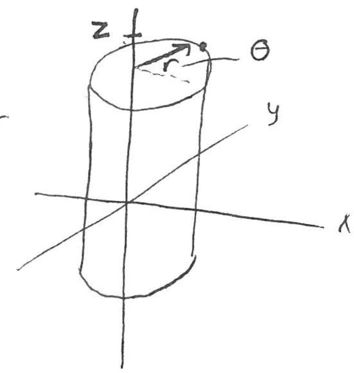
Lecture 10 25 july 2014

Cylindrical Coords

Spherical Coordinates

Cylindrical Coordinates

3d

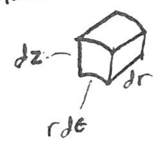


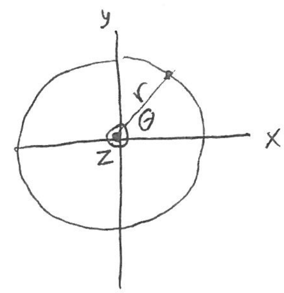
(x, y, z) can be written as (r, θ, z)

where $(x, y) = (r\cos\theta, r\sin\theta)$ as in polar

Volume element

$dV = r\,dr\,d\theta\,dz$

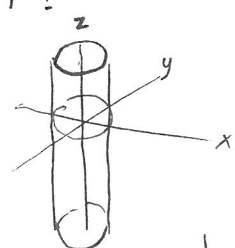




Example:

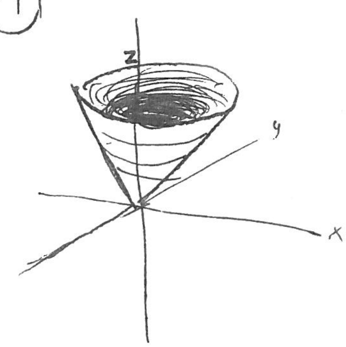
What is shape given by $0 \le r \le 1$?

Infinite cylinder



What is shape given by $z = r$?

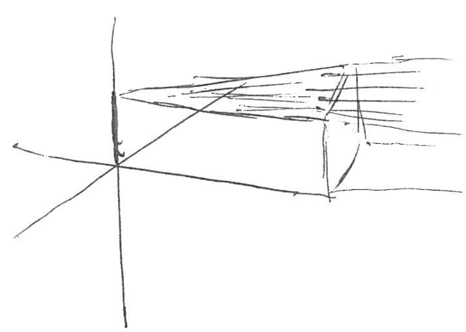
Cone



What is shape given by $0 \le \theta \le \pi/3$

$0 \le z \le 1$

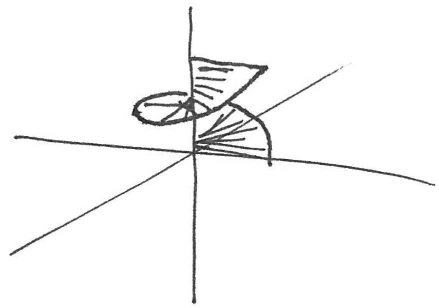
Wedge (infinite)
Slab



Example:

What shape is given by $z = \theta$ for $0 \le \theta < 2\pi$

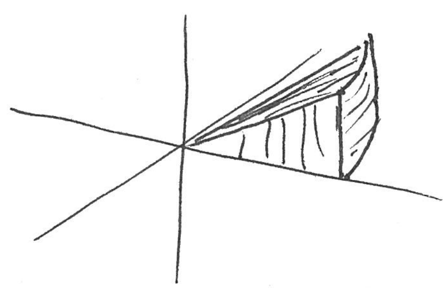
$r \le 1$



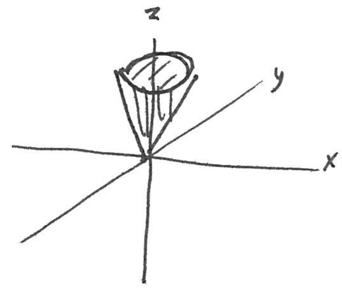
Spiral staircase

What shape is given by $0 \le \theta \le \pi/4$

$0 \le z \le r$

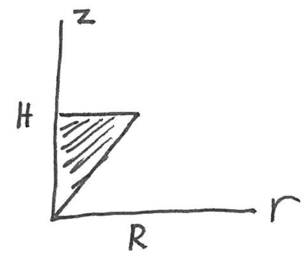


Example: Find Volume of cone of height H and radius R



Specify shape:

$$0 \le \theta \le 2\pi$$
$$0 \le z \le H$$
$$0 \le r \le \frac{Rz}{H}$$



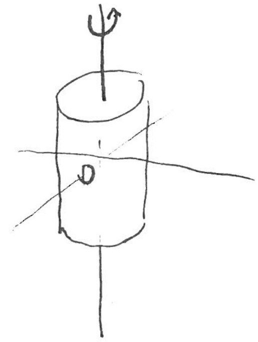
Write integral

$$V = \iiint_D dV$$
$$= \iiint_D r\,dr\,d\theta\,dz$$
$$= \int_{\theta=0}^{2\pi} \int_{z=0}^{H} \int_{r=0}^{\frac{R}{H}z} r\,dr\,dz\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_{z=0}^{H} \int_{r=0}^{\frac{R}{H}z} r\,dr\,dz$$
$$= 2\pi \int_0^H \frac{1}{2}\left(\frac{R}{H}\right)^2 z^2\,dz$$
$$= \frac{\cancel{2}\pi}{\cancel{2}}\left(\frac{R}{H}\right)^2 \frac{1}{3}H^3$$
$$= \frac{1}{3}\pi R^2 H.$$

Example:

Moment of inertia of cylinder radius R, length L. about axis of symmetry

Const. density ρ.



$$I = \iiint_D \rho r^2 \, r\,dr\,d\theta\,dz$$

Describe shape

$0 \le \theta \le 2\pi$

$0 \le z \le L$

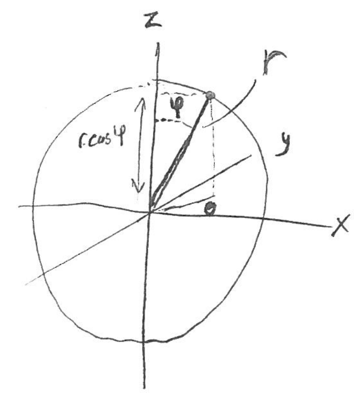
$0 \le r \le R$

$$I = \int_0^{2\pi} \int_0^{L} \int_0^{R} \rho r^3 \, dr\, d\theta\, dz$$

$$= \rho \int_0^{2\pi} d\theta \int_0^L dz \int_0^R r^3 dr$$

$$= \rho\, 2\pi\, L\, \frac{1}{4} R^4 = \frac{1}{2} (\rho \pi R^2 L) R^2 = \frac{1}{2} M R^2$$

Spherical Coordinates

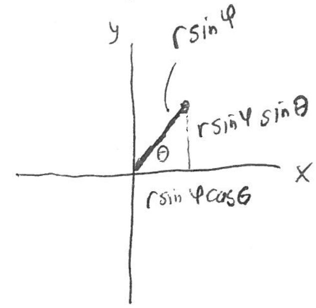


(x,y,z) can be written as (r,θ,φ)

r - distance to origin

φ - polar angle (angle from pole) (~latitude)

θ - azimuthal angle (angle about polar axis)



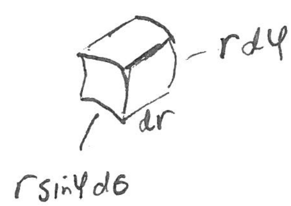
Caution: Physicists use θ for polar angle and φ for azimuthal angle!

$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi.$

Note $x^2+y^2+z^2=r^2$



Volume element

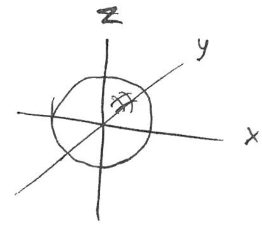
$$dV = r^2\sin\varphi\, dr\, d\varphi\, d\theta$$

polar angle

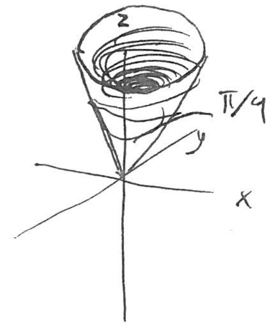
Note $0 \leq \varphi \leq \pi$

$0 \leq \theta < 2\pi$

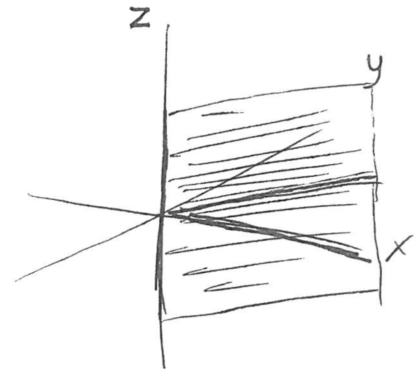
Example $r=1$ describes a sphere



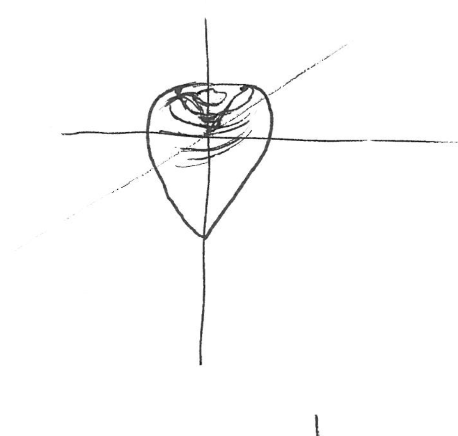
Example: $\varphi = \pi/4$ describes a cone



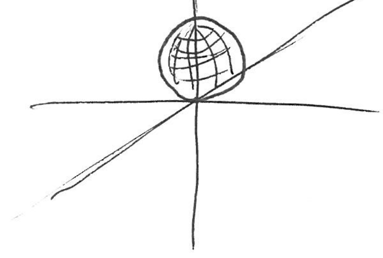
Example $\theta = \pi/4$ describes a half plane

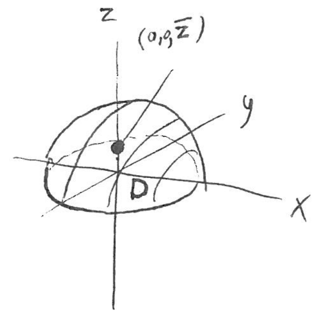


Example $\rho = \cancel{\sin}\,\varphi$



$\rho = \overset{\cos}{\cancel{\sin}}\,\varphi$ for $0 \le \varphi \le \pi/2$





Example: Find center of mass of hemisphere of radius R.

$$\bar{z} = \frac{\iiint_D z \rho \, dV}{\iiint_D \rho \, dV} = \frac{\iiint_D z \, dV}{\iiint_D dV}$$

Note $\iiint_D dV =$ Volume of $D = \frac{2}{3}\pi R^3$

$$\iiint_D z \, dV = \; ?$$

Describe D in sperical coordinates

$$0 \leq r \leq R$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq \varphi \leq \pi/2$$

$$\int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/2} \int_{r=0}^{R} r\cos\varphi \; r^2 \sin\varphi \, dr \, d\varphi \, d\theta$$

$$\int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\varphi \cos\varphi \, d\varphi \int_0^R r^3 \, dr$$

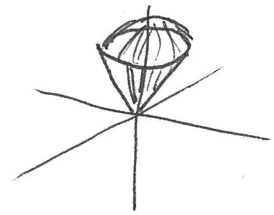
$$I = 2\pi \quad \tfrac{1}{2}\sin^2\varphi \Big|_0^{\pi/2} \quad \tfrac{1}{4}R^4$$

$$= 2\pi \left(\tfrac{1}{2}\right) \tfrac{1}{4} R^4 = \frac{\pi R^4}{4}$$

So $\bar{z} = \frac{3}{8} R$

Activity: Find the volume of the part of unit sphere with $\varphi \leq \pi/4$

a) Draw picture



b) Set up integral

$$V = \iiint dV$$

$$= \int_{\varphi=0}^{\pi/4} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^2 \sin\varphi \, dr\, d\theta\, d\varphi$$

$$= \int_0^{\pi/4} \sin\varphi \, d\varphi \int_0^{2\pi} d\theta \int_0^1 r^2 dr$$

$$= -\cos\varphi \Big|_0^{\pi/4} \quad 2\pi \quad \frac{1}{3}$$

$$= \left(1 - \frac{1}{\sqrt{2}}\right) \frac{2\pi}{3}.$$