1 August 2013
Calculus 3, Interphase 2013
Paul E. Hand
hand@math.mit.edu

Problem Set 5

Due: **Monday 5 August 2013** in class.

1. (20 points) *Gaussians and Change of Variables*

   (a) Let $\rho(t, x, y, z)$ be the concentration of a chemical at time t and position (x, y, z). If one unit of chemical is localized to $(0, 0, 0)$ at $t = 0$ and is permitted to diffuse with diffusion constant D, the concentration obeys

   $$\rho(t, x, y, z) = \frac{1}{(4\pi D t)^{3/2}} \exp\left(-\frac{x^2 + y^2 + z^2}{4Dt}\right).$$

   An important physical property in such systems is that mass is conserved. Show that

   $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(t, x, y, z) dx dy dz$$

   is constant in time by changing variables to $(\tilde{x}, \tilde{y}, \tilde{z}) = (x, y, z)/\sqrt{t}$.

   (b) By an appropriate change of variables, find the value of c such that

   $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c \exp\left(-\frac{(x+y)^2}{2\sigma_1^2} - \frac{(x-y)^2}{2\sigma_2^2}\right) dx dy = 1$$

2. (20 points) Sketch the following vector fields. Are they conservative? If so, find a potential function.

   (a) $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$

   (b) $\mathbf{F}(x, y) = \langle y - x, x - y \rangle$.

   (c) $\mathbf{F}(x, y) = \frac{\langle 1-y, x-2 \rangle}{\sqrt{(x-2)^2 + (y-1)^2}}$

   Hint: How are the vectors $\langle a, b \rangle$ and $\langle -b, a \rangle$ related? Also, what would integrals over closed curves be if it is conservative?

3. (20 points) Let $\mathbf{F}(x, y) = \langle \cos y, -x \sin y \rangle$. Let C be the line segment connection $(0, 0)$ to $(2, \pi)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$

   (a) Directly

   Feel free to look up any antiderivatives.

   (b) By the fundamental theorem

4. (20 points) *Area of a polygon*

   (a) Use Green's theorem to show that the area inside the closed, positively oriented curve C is given by both $\oint_C x dy$ and $-\oint_C y dx$.

   (b) Suppose the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ specify a positively oriented polygon. By computing one of the line integrals from (a), show that the area of the polygon is

$$\sum_{i=1}^{n} \frac{x_{i+1} + x_i}{2}(y_{i+1} - y_i),$$

   where we interpret (x_{n+1}, y_{n+1}) to mean (x_1, y_1).

5. (20 points) *Particle trapping by electrostatic forces.* Away from point charges, the 2d electric potential $\phi(x, y)$ satisfies

$$\partial_{xx}\phi(x, y) + \partial_{yy}\phi(x, y) = 0 \qquad (*)$$

   If $(0, 0)$ is a local minimum of ϕ, then a particle located at $(0, 0)$ will be trapped in a stable manner. We will prove that this is impossible. Our approach is to show that any function ϕ satisfying (*) obeys the mean value property: $\phi(0, 0)$ equals the average value of ϕ over any circle centered at the origin. Hence, unless ϕ is constant, ϕ takes values both greater than and less than $\phi(0, 0)$ along every circle centered at the origin. Hence $\phi(0, 0)$ can not be a local minimizer.

   (a) Let C be the positively oriented unit circle, centered at the origin. The line integral $f(r) = \frac{1}{2\pi}\int_C \phi(rx, ry)ds$ is the average value of ϕ on the circle of radius r centered at the origin. Compute $f'(r)$ and express it in the form $\frac{1}{2\pi}\int_C \mathbf{F}(x, y) \cdot \langle -y, x\rangle ds$.

   (b) Because the unit tangent vector to C is $\langle -y, x\rangle$, we can write $f'(r) = \frac{1}{2\pi}\int_C \mathbf{F} \cdot \langle dx, dy\rangle$. Use Green's theorem to show that f is constant. Hence, continuity of ϕ guarantees the average value of ϕ over any circle centered at the origin equals the value $\phi(0, 0)$.

   (c) Could we have used the second derivative test to conclude that ϕ has no local minimizers?