16 July 2013
Calculus 3, Interphase 2013
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Problem Set 3

Due: **Monday 22 June 2013** in class.

1. Consider the ellipsoid defined by $x^2 + 2y^2 + 3z^2 = 9$.

   (a) Find the tangent plane at the point $(2, -1, 1)$ by viewing the surface as a level set of a function of three variables.
   Hint: How are gradients and level sets related?

   (b) Use the plane from (a) to approximate the value of z on the surface at the coordinates $x = 2.01$, $y = -0.99$. Compute the exact value of z and determine how much error is in the approximation.

2. Consider a hill with height given by $z = x^2 - y^2$. Suppose a hiker is initially at $(2, 1, 3)$ and always walks in the direction of steepest ascent.

   (a) Sketch the level sets of z and the path taken by the hiker.

   (b) Initially, what is the 3d vector tangent to the hiker's path?

   (c) Ignoring z, find the curve in (x, y) traced out by the hiker.

3. Suppose $w = f(x, y)$ and that $x(r, \theta) = r\cos\theta$ and $y(r, \theta) = r\sin\theta$. The function w can be viewed as function of (x, y) or it can be viewed as a function of (r, θ). Show that

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$$

4. Find all critical points of $f(x, y) = x^4 + y^4 - 4xy$. Classify them as a local max, a local min, or a saddle point.

5. *Best fit line*. Consider the points $(0, 1), (1, 1), (2, 2), (3, 3)$. The residual of the point (x_i, y_i) with the line $y(x) = mx + b$ is defined as $y_i - y(x_i)$. The best-fit line is defined to be the line that minimizes the average of the squared residuals (aka the mean square error). Find m and b of the best fit line.

6. Use Lagrange multipliers to find radius and the height of the cylinder with surface area S that has maximal volume.

7. Consider a current I going through three resistors in parallel. If each has resistance R_1, R_2, R_3, and the respective current through each resistor is I_1, I_2, I_3, then the power dissipated by each resistor is $I_1^2 R_1$, $I_2^2 R_2$, $I_3^2 R_3$. Note that $I = I_1 + I_2 + I_3$. The current splits up in a way that minimizes the total power dissipated by the resistors.

   (a) Using the constraint to remove one variable, show that the currents that minimize the total power dissipation are such that $I_1 R_1 = I_2 R_2 = I_3 R_3$.

   (b) Use Lagrange Multipliers to show the same thing.