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Calculus 3, Interphase 2013
Paul E. Hand
hand@math.mit.edu

Problem Set 2 [Revised]

Due: **17 June 2013** in class.

1. (10 points) This problem concerns finding the line that is the intersection of the planes $x + y + z = 1$ and $x - y - z = 2$.

   (a) Using the first equation, we can write that $x = 1 - y - z$. Plugging into the second equation, we get $-2y - 2z = 1$. Can we conclude that $-2y - 2z = 1$ is the desired line? If not, what is its relationship to the desired line?

   (b) Find the desired line in parametric form.

2. (10 points)

   (a) Let $\mathbf{a}(t)$ and $\mathbf{b}(t)$ give curves in 3d. By writing out the components explicitly, show that

   $$\frac{d}{dt}\Big(\mathbf{a}(t) \cdot \mathbf{b}(t)\Big) = \frac{d\mathbf{a}}{dt} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{dt}$$

   (b) Show that if the speed of an object is constant, then its acceleration is always perpendicular to its velocity.
   Hint: If the object has velocity $\mathbf{v}(t)$, study $\frac{d}{dt}|\mathbf{v}(t)|^2$.

3. (20 points) Parameterizations

   (a) Parameterize the square traversed from $(0, 0)$ to $(0, 1)$ to $(1, 1)$ to $(1, 0)$ and back to $(0, 0)$. Your answer should be expressed as a piecewise linear function $\mathbf{X}(t)$.

   (b) A gear of radius R is centered at the origin and fixed so that it can not move or rotate. Another gear of radius r initially touches the fixed gear at $(-R, 0)$ and rotates clockwise around the fixed gear. Sketch and find the trajectory of the point originally touching the fixed gear.

4. (15 points) Let $\mathbf{X}(t) = (e^{-t} \cos t, e^{-t} \sin t)$ for $0 \leq t < \infty$.

   (a) Plot this curve.

   (b) Compute the velocity vector and speed as functions of t.

   (c) Compute the length of the curve.

   (d) How many times does the curve circle the origin?

   (e) Does the contrast of (c) and (d) surprise you?

   (f) Extra credit (5 points): Find a curve of the form $(r(t) \cos t, r(t) \sin t)$ that spirals into the origin and has infinite arc length. Justify your answer with a calculation.

5. (30 points) Solutions to partial differential equations

   (a) Let $u(t,x)$ be the pressure of air at time t and 1d position x. Pressure disturbances represent sound and travel according to the wave equation:

   $$\partial_{tt}u(t,x) - c^2\partial_{xx}u(t,x) = 0.$$

   Show that $u(t,x) = \sin(x - ct)$ satisfies the wave equation. This function corresponds to a pure tone of what frequency? Assuming $c > 0$, in which direction is the wave moving? At what speed?

   (b) Let $\phi(t,x)$ be the electric potential at time t and position x along the axon of a neuron. The electrical excitation of the neuron travels according to the cable equation:

   $$\partial_t\phi(t,x) = \partial_{xx}\phi(t,x) + f(\phi(t,x)),$$

   where the function f is given by the membrane ion channel dynamics. Show that $\phi(t,x) = g(x - ct)$ is a solution to the cable equation if the function $g(\xi)$ satisfies $g''(\xi) + cg'(\xi) + f(g(\xi)) = 0$ for any ξ. If we could find a g solving this relationship, then we would have a theoretical computation for conduction speed that could be experimentally checked.

   (c) Let $T(t,x)$ be the temperature at time t and position x along a long metal rod. If the rod is initially heated at the origin, the heat will spread in both directions according to the heat equation:

   $$\partial_t T(t,x) - \partial_{xx}T(t,x) = 0.$$

   Show that $T(t,x) = \frac{1}{\sqrt{4\pi t}}\exp(-\frac{x^2}{4t})$ satisfies the heat equation for any $t > 0$ and $-\infty < x < \infty$. Sketch the function T at various times t. The x^2/t term in the exponent reveals that in time t, heat travels a distance proportional to $\sqrt{t}$. That is to say, to get the heat to travel twice as far, you have to wait four times as long!

6. (15 points) Sketch the level curves and the three-dimensional surfaces given by

   (a) The saddle: $f(x,y) = x^2 - y^2$

   (b) The 2d bell curve: $f(x,y) = e^{-x^2-y^2}$

   (c) The gravitational potential of a 2d planet: $f(x,y) = \log\sqrt{x^2+y^2}$