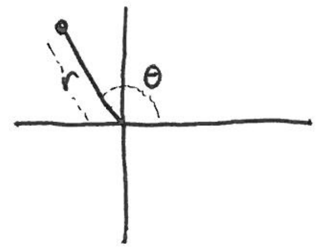
Lecture 9 22 july 2013

Polar Coordinates

Double Integrals in Polar

Polar Coordinates



A point in 2d can be described by the polar coordinates (r, θ)

where $r \geq 0$ and $0 \leq \theta < 2\pi$
or
$-\pi < \theta \leq \pi$

Converting from polar to cartesian:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Converting from cartesian to polar

$$r = \sqrt{x^2+y^2}$$

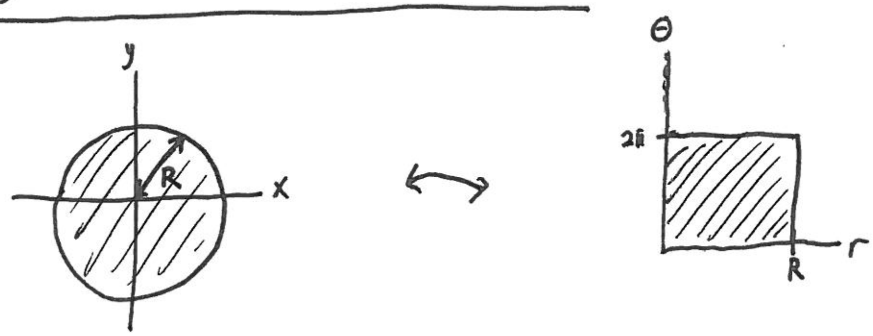
$$\theta = \underbrace{\tan^{-1} y/x}_{\text{cant distinguish Quadrants I/III or II/IV}} \quad \text{or} \quad \tan^{-1}(x, y)$$

← annoying to write b/c you need to specify which 2π range is at pt.
y/x causes divide by 0 issues

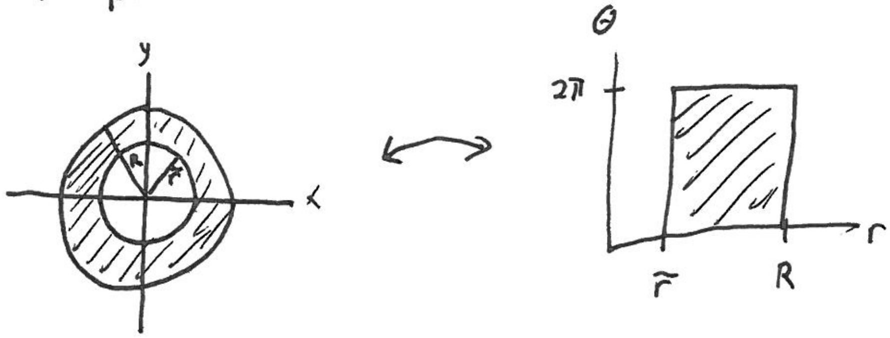
Note: The origin is represented by many values of (r, θ)

$r = 0 \quad \theta = \text{anything!}$

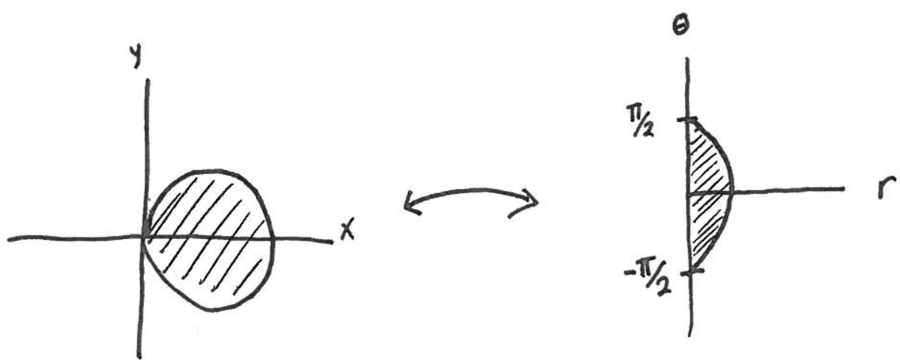
Regions in Polar & Cartesian

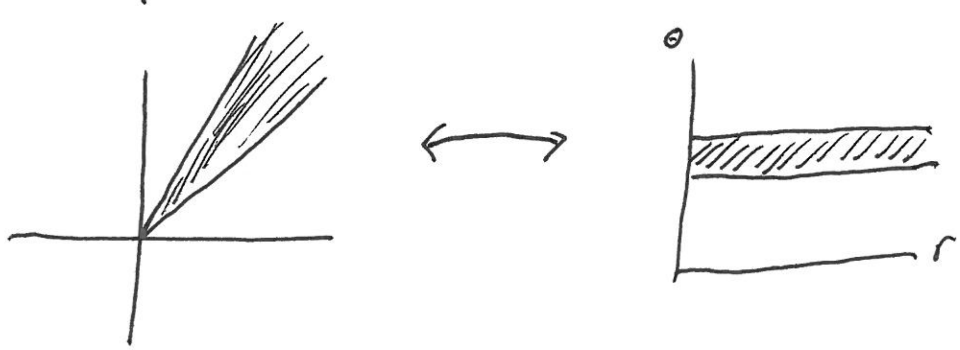


A circle in cartesian coordinates is a rectangle in polar coordinates



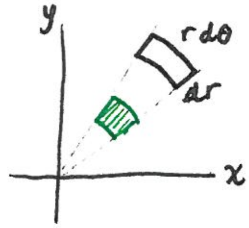
An annulus in cartesian coordinates is also rectangle in polar



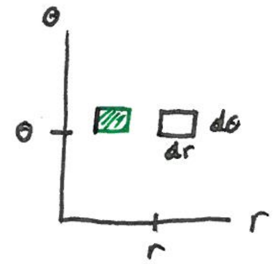


Area Element in Polar

A region at (r,θ) of size $dr \times d\theta$ has area



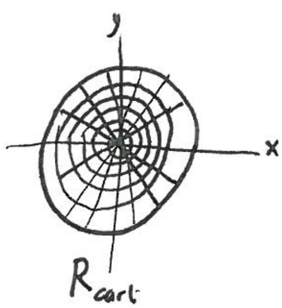
$\longleftrightarrow$



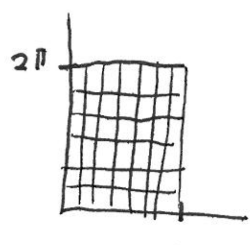
$$dA = r\,dr\,d\theta$$

- Region has area in $r\theta$ plane different from area in xy plane
- These two regions (of same size in $r\theta$) have different areas in xy plane

So $$\iint_{R_{cart}} f(x,y)\,dx\,dy = \iint_{R_{polar}} f(r,\theta)\,r\,dr\,d\theta$$

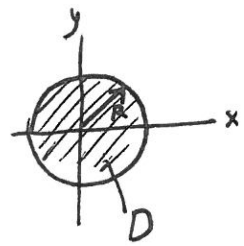


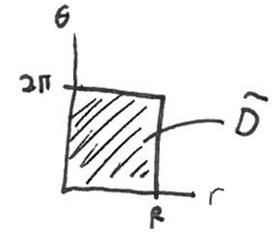
$\rightarrow$



Adding up all these areas is like adding up all these areas weighted by r

Example: Area of circle of radius R





$$\iint_D 1 \, dx\,dy = \iint_{\tilde{D}} 1 \; r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{R} 1 \; r\,dr \; d\theta$$

— can separate b/c r integral constant w.r.t. θ

$$= \int_{\theta=0}^{2\pi} d\theta \int_{r=0}^{R} r\,dr$$

$$= 2\pi \; \tfrac{1}{2}R^2 = \pi r^2$$

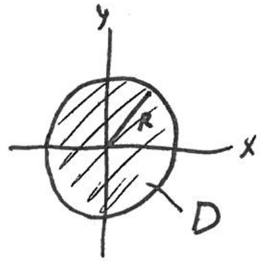
Expressing a double integral in Polar

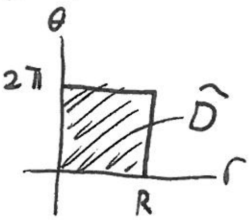
To evaluate $\iint_R f(x,y)\,dx\,dy$

- Express R in polar coordinates
  - θ ranges from θ_{min} to θ_{max}, $r(\theta)$ ranges from $r_{min}(\theta)$ to $r_{max}(\theta)$

  OR
  - r ranges from r_{min} to r_{max}, $\theta(r)$ ranges from $\theta_{min}(r)$ to $\theta_{max}(r)$
- Express f in terms of r & θ
- Express area element in polar $dx\,dy = r\,dr\,d\theta$
- Evaluate double integral as normal.

Example: Moment of inertia of disk of radius R, area-mass density ρ (constant)





$$\iint_D (x^2+y^2)\,\rho\,dx\,dy = \iint_{\tilde{D}} r^2 \rho\; r\,dr\,d\theta$$

$$= \rho \int_{\theta=0}^{2\pi} \int_{r=0}^{R} r^3\,dr\,d\theta$$

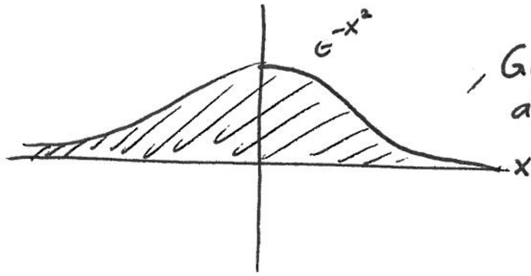
$$= \rho \int_{\theta=0}^{2\pi} d\theta \int_{r=0}^{R} r^3\,dr$$

$$= \rho\, 2\pi\, \frac{1}{4} R^4 = \rho \frac{\pi}{2} R^4 = \frac{1}{2}(\pi R^2 \rho) R^2$$

$$= \frac{1}{2} M R^2$$

IMPORTANT Example

Show $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$



Gaussian
aka Bell curve

Most important
probability distribution

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$ cant solve by substitution or guessing antiderivative

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$$

$$= \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2}\,r\,dr$$

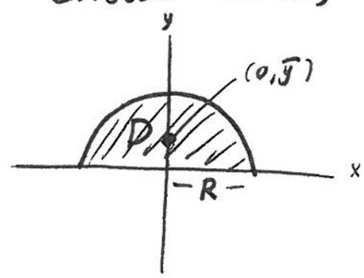
$$= 2\pi \left.\frac{e^{-r^2}}{-2}\right|_0^{\infty}$$

$$= \pi$$

So $\boxed{I = \sqrt{\pi}}$!!

Example:

Find center of mass of half of disk of constant density



$\bar{X} = 0$ by symmetry

$$\bar{y} = \frac{\iint_D y \, \rho \, dx\,dy}{\iint_D \rho \, dx\,dy}$$

Evaluate: $\iint_D \rho \, dx\,dy = \rho \iint_D dx\,dy = \rho \cdot \text{Area of } D$

$$= \rho \, \tfrac{1}{2} \pi R^2$$

$$\iint_D y \, \rho \, dx\,dy = \int_{\theta=0}^{\pi}\int_{r=0}^{R} r \sin\theta \, \rho \, r \, dr \, d\theta$$

$$= \rho \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{r=0}^{R} r^2 \, dr$$

$$= \rho \left[-\cos\pi\right|_0^{\pi} \cdot \tfrac{1}{3} R^3$$

$$= \rho \, 2 \, \tfrac{1}{3} R^3$$

So $\bar{y} = \dfrac{\frac{2}{3} \cancel{\rho} R^3}{\cancel{\rho} \frac{1}{2} \cancel{\pi} R^2} = \dfrac{4}{3\pi} R = \dfrac{4R}{3\pi}$