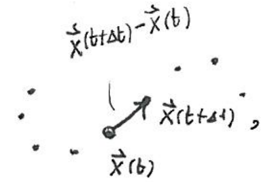
Lecture 5 12 july 2013

Velocity, Tangent Curves, Arc Length
Functions of several variables
Partial Derivatives

Why is velocity tangent to curve?

Let $\vec{X}(t)$ be a curve

$$\vec{V}(t) = \frac{d\vec{X}(t)}{dt}$$



$$= \lim_{\Delta t \to 0} \frac{\vec{X}(t+\Delta t) - \vec{X}(t)}{\Delta t}$$

$$\approx \frac{\vec{X}(t+\Delta t) - \vec{X}(t)}{\Delta t}$$ which is in direction along curve

Arclength

To find length of a parameterized curve, integrate speed.

Arc length of $\vec{X}(t)$ from $t=a$ to b is

$$S = \int_a^b \left| \frac{d\vec{X}(t)}{dt} \right| dt$$

Example: Perimeter of circle radius r.

Parameterize $\vec{X}(t) = (r\cos wt, r\sin wt)$ $\quad 0 \leq t \leq 2\pi/w$

$$\frac{d\vec{X}}{dt} = +rw(-\sin wt, \cos wt) \qquad \left|\frac{d\vec{X}}{dt}\right| = rw$$

$$S = \int_0^{2\pi/w} rw\,dt = rw\,\frac{2\pi}{w} = 2\pi r.$$

Note: arc length formula works for any parameter (not just for time)

~~Example~~

$$S = \int_a^b \left| \frac{d\vec{X}}{d\theta}(\theta) \right| d\theta$$

~~Example:~~

Functions of several variables

- $f(x,y)$ or $f(x,y,z)$ or $f(\vec{x})$
- Takes x&y or (x,y) as input and returns a number
- Three main ways of visualizing:
    - Level sets of $f(x,y)$
    - 3d surfaces drawn in perspective : $(z = f(x,y))$
    - Sequence of cross sections (especially if one var. is time)

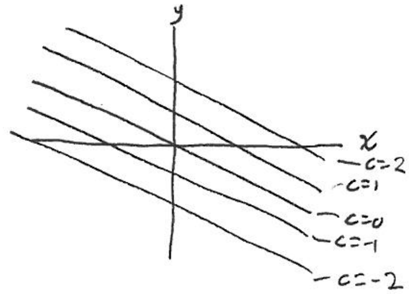
Visualizing by drawing level sets

- The level sets of $f(x,y)$ are the curves along which f is constant.
- The c-level set is the curve along which $f(x,y)=c$

eg. Find level sets of $x+2y$.

$x+2y=c$ is a line with normal vector $(1,2)$

Different c's correspond to different lines



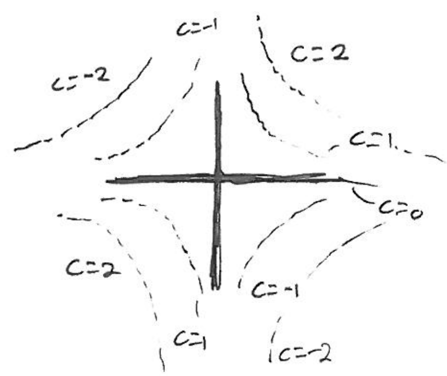
Note: surface given by $z=x+2y$ is a plane

eg. Find level sets of xy

$xy=c$

If $c\neq 0$, $y=c/x$

If $c=0$, $y=0$ or $x=0$



To find level sets:

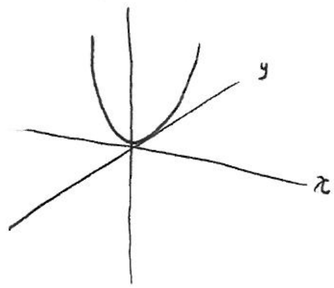
- First look for 0-level set. What curve given by ~~Plot~~ $f(x,y)=0$? Solve for $y(x)$ if you must.
- Look for positive level sets
- Look for negative level sets

Visualizing by drawing 3d surface

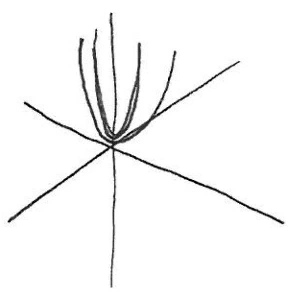
To Plot surface $Z = f(x,y)$:

- Plot cross sections for values of x
- Plot cross sections for values of y
- Connect. Pay attention to level sets

Ex: plot $Z = x^2 + y^2$

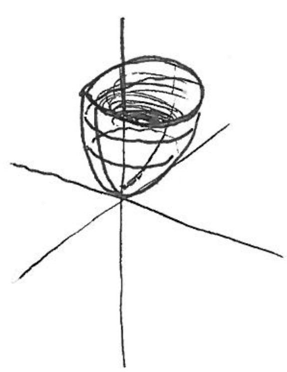


$y=0$ cross section
$Z = x^2$



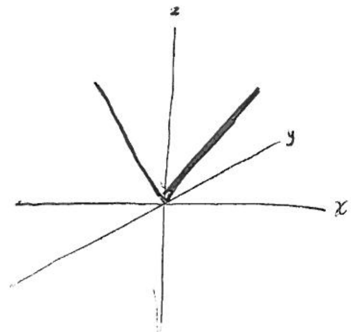
$x=0$ cross section
$Z = y^2$

Note: level sets of x^2+y^2 are circles.

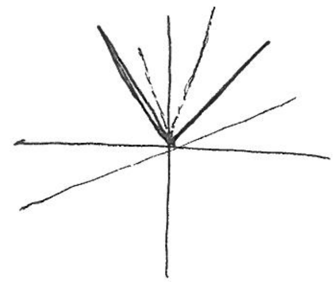


Ex: Plot $z = |x| + |y|$

Plot $y=0$ cross section



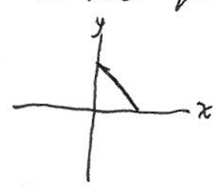
Plot $x=0$ cross section



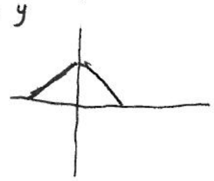
Note: level sets of $|x|+|y|$ are diamonds

why: $|x|+|y| = c$. This shape is mirror symmetric about $x=0$, & about $y=0$
plot in first quadrant:

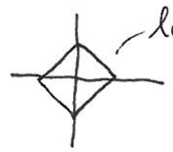
If $x \geq 0, y \geq 0$ $x + y = c$



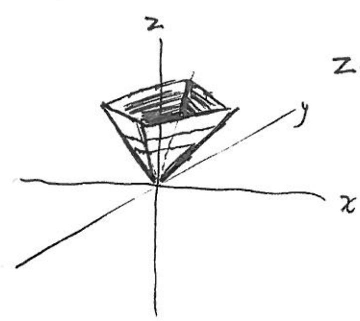
Enforce symmetry wrt y



Enforce sym wrt x



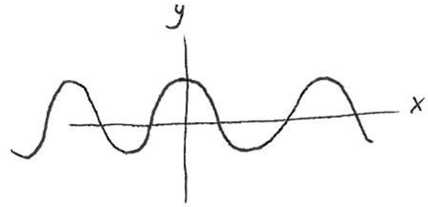
Connect 3D sketch



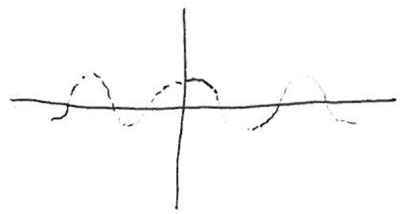
Visualizing by sequence of sketches

Visualize $y = e^{-t} \cos x$

At $t=0$

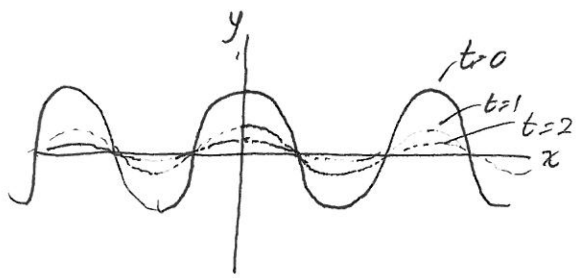


At $t=1$



And so on

Combine



Partial Derivatives

with respect to

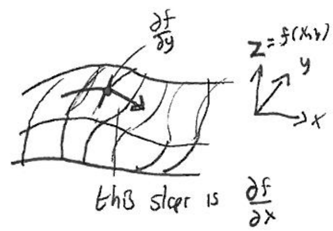
The partial derivative of $f(x,y)$ wrt x is the normal derivative where y is treated as a constant

Similarly for partial deriv wrt y.

Notation: $\frac{\partial f}{\partial x}(x,y)$ & $\partial_x f(x,y)$ & $f_x(x,y)$

Precisely: $\frac{\partial f}{\partial x}(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$

~~Visually:~~
Geometrically: slope in x direction



Computationally: Just ignore the other variables

Higher derivatives: $\frac{\partial^2 f}{\partial x^2}(x,y)$ $\frac{\partial^2 f}{\partial x \partial y}(x,y)$ deriv first in y, then in x. (but doesnt matter order)

$\partial_{xx} f(x,y)$ $\partial_{xy} f(x,y)$

$f_{xx}(x,y)$ $f_{yx}(x,y)$

Example: $f(x,y) = x^2 + 2xy + y^2$

$f_x(x,y) = 2x + 2y$ $\quad f_y(x,y) = 2x + 2y$

$f_{xx}(x,y) = 2$ $\quad f_{yx}(x,y) = 2$

$f_{xxx}(x,y) = 0$ $\quad f_{yy}(x,y) = 2$

Example: $f(x,y) = \log(x + e^y)$. ~~Assume $x > 0$, Compute~~

$\partial_x f(x,y) = \dfrac{1}{|x + e^y|}$ $\quad \partial_y f(x,y) = \dfrac{e^y}{|x+e^y|}$

Example: Show that $\phi(x,y) = \log(x^2+y^2)$ satisfies $\partial_{xx}\phi(x,y) + \partial_{yy}\phi(x,y) = 0$

Compute $\partial_x \phi$, $\partial_y \phi$, $\partial_{xx}\phi$, $\partial_{yy}\phi$ and show equality satisfied.

$\phi(x,y) = \log(x^2+y^2)$

$\partial_x \phi(x,y) = \dfrac{2x}{x^2+y^2}$ $\quad \partial_y \phi = \dfrac{2y}{x^2+y^2}$

$\partial_{xx}\phi(x,y) = \dfrac{2}{x^2+y^2} + \dfrac{2x(-1)2x}{(x^2+y^2)^2}$ $\quad \partial_{yy}\phi(x,y) = \dfrac{2}{x^2+y^2} - \dfrac{4y^2}{(x^2+y^2)^2}$

$= \dfrac{2}{x^2+y^2} - \dfrac{4x^2}{(x^2+y^2)^2}$

So $\partial_{xx}\phi(x,y) + \partial_{yy}\phi(x,y) = \dfrac{2}{x^2+y^2} - \dfrac{4x^2}{(x^2+y^2)^2} + \dfrac{2}{x^2+y^2} - \dfrac{4y^2}{(x^2+y^2)^2}$

$= \dfrac{4}{x^2+y^2} - 4\dfrac{(x^2+y^2)}{(x^2+y^2)^2} = 0$ ✓