- Review of Series
  - Sequence: a list of numbers in a particular order
    - Infinite sequence: an infinite list of numbers in a particular order
  - Example:

$$\{a_n\} = \{a_1, a_2, a_3, a_4 \ldots\}$$

1st term   2nd term   ...

  - Series: the sum of all the terms of a sequence
    - Infinite series: the sum of the terms in an infinite sequence
    - Example:

end at infinite term

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \ldots$$

start at 1st term

    - Specific Types of Series
      - Geometric Series: a series in which each term is obtained from the previous one by **multiplying** it by the **common ratio** r

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} = \sum_{n=1}^{\infty} ar^{n-1} \quad (a \neq 0)$$

common ratio: r

      - Alternating Series: a series whose terms are alternately positive and negative
        - Example: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$
      - Power Series:

$$1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

not n=1!!!

      - P-Series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \text{ where } p \geq 0$$

        - Example:

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

p=3

        - When p=1: Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

- Convergence and Divergence
  - Convergence: approaches a definite, real number as you go off to infinity

$$S_n=\sum_{n=1}^{\infty} a_n \rightarrow \text{Converges if } \lim_{n\to\infty} S_n = \text{a real number}$$

  - Divergence: doesn't approach a definite, real number as you go off to infinity
  - How to Tell if a Series Converges or Diverges: Using Convergence Tests
    - We can use convergence tests to analyze series' end behavior
    - Test for Divergence (looking at limits)
      - If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. If $\lim_{n\to\infty} a_n$ doesn't exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent
      - Example:

$$\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4} \quad \text{Does this series converge or diverge?}$$

$$\lim_{n\to\infty}\left(\frac{n^2}{5n^2+4}\right) = \lim_{n\to\infty}\left(\frac{n^2}{5n^2+4}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right) = \lim_{n\to\infty}\left(\frac{1}{5+\frac{4}{n^2}}\right) = \frac{1}{5} \neq 0$$

$\Rightarrow$ Diverges by the Test for Divergence

    - The Integral Test
      - Suppose f is a continuous, positive, decreasing function on the interval $[1,\infty)$ and let $a_n= f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

Aka:

(a) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent

(b) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent

*Note: When we use the Integral Test it's not necessary to start the series or integral at n=1. Example:

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \rightarrow \int_4^{\infty} \frac{1}{(x-3)^2}dx$$

      - Example:

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n} \quad \text{Does this series converge or diverge?}$$

$$\text{let } f(x) = \frac{\ln(x)}{x}$$

1. Check to see if $f(x)$ is ultimately decreasing

$$f(x) = \frac{\ln(x)}{x}$$

$$\Rightarrow f'(x) = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

$f'(x) < 0$ when $\ln(x) > 1 \Rightarrow f(x)$ is decreasing

$\Rightarrow$ Use the Integral Test

2. Apply Integral Test

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln(x)}{x}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{(\ln(x))^2}{2}\right]_1^t = \lim_{t\to\infty} \frac{(\ln(t))^2}{2} = \infty$$

3. Analyze Results

$\int_1^\infty f(x)\,dx = \infty \Rightarrow$ The improper integral is divergent

$\Rightarrow$ By the Integral Test, $\sum\limits_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges

- The Comparison Test
  - Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms
    - If $\sum b_n$ is convergent and $a_n \le b_n$ for large enough n, then $\sum a_n$ is also convergent
    - If $\sum b_n$ is divergent and $a_n \ge b_n$ for large enough n, then $\sum a_n$ is also divergent
  - Example:

$$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$$

Given that $\frac{5}{2}\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, does this series converge or diverge?

1. Recognize the relationship between the two series

$$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} < \frac{5}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$$

(because the left side has a bigger denominator)

② Apply the Comparison Test

Since $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} < \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$

and $\frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent,

then $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges by the Comparison Test

- Specific Case: P-Series Convergence
  - P-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{, where } p \geq 0$$

- For the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p>1$ and divergent if $p \leq 1$
- Example:

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ Does this series converge or diverge?

$p = 2 \Rightarrow p > 1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent

- Specific Case: Geometric Series Convergence
  - Geometric Series:

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} = \sum_{n=1}^{\infty} ar^{n-1} \quad (a \neq 0)$$

common ratio : r

- A geometric series converges if $|r|<1$ and diverges if $|r| \geq 1$
- Example:

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$ Does this series converge or diverge?

$|r| = \frac{1}{3} \Rightarrow |r| < 1$

$\Rightarrow \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$ converges

- Related Case: Exponential Convergence
  - Form: $\sum_{n=1}^{\infty} e^{kn}$

- If k<0 then the series converges
- If k≥0 then the series diverges

○ Example:

$$\sum_{n=1}^{\infty} e^{5n}$$ Does this series converge or diverge?

$$k=5 \Rightarrow k \geq 0$$

$$\Rightarrow \sum_{n=1}^{\infty} e^{5n} \text{ diverges}$$

- The Ratio Test
  - If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent
  - If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent
  - If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum_{n=1}^{\infty} a_n$
  - Example:

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$$ Does this series converge or diverge?

$$a_n = \frac{(-1)^n n^3}{3^n}$$

$$a_{n+1} = \frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}}\right|$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}\right] = \lim_{n\to\infty}\left[\frac{1}{3} \cdot \frac{(n+1)^3}{n^3}\right] = \frac{1}{3}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3$$

$$= \frac{1}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3} < 1$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1, \text{ so } \sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n} \text{ is convergent by the Ratio Test}$$

- Taylor Series
  - Basic idea: rewrite any function in terms of a polynomial
  - Form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \ldots$$

not n=1!!!

* Taylor series of a function of f centered at a

* a can be any real number

  - Special Case: Maclaurin Series
    - Taylor series centered at a=0
    - Form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots$$

    - Common Maclaurin Series Expansions
      - $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$
      - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
      - $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$
      - $\cos x = \sum_{n=(0)}^{\infty} (-1)^n \frac{x^{2n}}{2n!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

  - Applications of Taylor Series
    - Linear approximations of functions
    - Higher order approximations
    - Make evaluating limits and integrals easier
  - Example: Derive the first 3 terms of MacLaurin Series of f(x)= e^x and use it to approximate f(0.5)

1 Find Taylor expansion centered at a=0

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots$$

$$f(x) = e^x \rightarrow f(0) = 1$$

$$f'(x) = e^x \rightarrow f'(0) = 1$$

$$f''(x) = e^x \rightarrow f''(0) = 1$$

$$\Rightarrow f(x) = e^x \approx 1 + \frac{x}{1!} + \frac{x^2}{2!}$$

(2) Use the Maclaurin series terms to find an approximation

$$f(0.5) = ?$$

$$f(x) = e^x \approx 1 + \frac{x}{1!} + \frac{x^2}{2!}$$

$$\Rightarrow f(0.5) = e^{0.5} \approx 1 + \frac{0.5}{1!} + \frac{(0.5)^2}{2!} = 1 + \frac{1}{2} + \frac{1}{8} = \frac{13}{8}$$

- Example:

Find $\lim\limits_{x \to 0} \left(\frac{1 - f(x)}{x^2} \right)$, where $f(x) = \cos(x^2)$

(1) Recall a Taylor series expansion of a similar form

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

(2) Use this known expansion to find the desired expansion

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\Rightarrow \cos x^2 = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \ldots$$

$$= 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \ldots$$

(3) Use the expansion to evaluate the limit

$$\lim_{x \to 0} \left(\frac{1 - f(x)}{x^2} \right) = \lim_{x \to 0} \left[\frac{1 - \left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \ldots\right)}{x^2} \right]$$

$$= \lim_{x \to 0} \left[\frac{\frac{x^4}{2!} - \frac{x^8}{4!} + \frac{x^{12}}{6!} - \ldots}{x^2} \right] = \lim_{x \to 0} \left[\frac{x^2}{2!} - \frac{x^6}{4!} + \frac{x^{10}}{6!} - \ldots \right] = 0$$