Day 7 2/4/2015

Nets

Sphere in $\mathbb{R}^n$ (S^{n-1}).

Want a finite set of points $\{x_i\}$ such that every $y \in S^{n-1}$ is near an x_i

$$\|y - x_i\|_2 < \varepsilon.$$

How many points do you need?

Is it polynomial in n? or exponential in n?

Such a ~~Let~~ ~~$\{x_i\}$~~ $N_\varepsilon = \{x_i\}$ is called an ε-net of the sphere.

Lemma: $\exists\, N_\varepsilon$ st $|N_\varepsilon| \leq \left(1 + \frac{2}{\varepsilon}\right)^n$

Proof: volume argument

Example $\exists\, N_{1/4}$ st $|N_{1/4}| \leq 9^n$

Spectral Norm of Symmetric Matrix via Nets

If $A \in \mathbb{R}^{n \times n}$ is symmetric

Rayleigh Quotient

$$\|A\| = \lambda_{max}(A) = \sigma_{max}(A) = \sup_{x \in S^{n-1}} |\langle Ax, x \rangle|$$

This sup is over ∞ly many points. Could check a net to get approximate bound.

Lemma: Let $\varepsilon \in (0, \frac{1}{2})$ & N_ε an ε-net

$$\|A\| \leq \frac{1}{1-2\varepsilon} \sup_{x \in N_\varepsilon} |\langle Ax, x \rangle|$$

Pf: Let $y \in S^{n-1}$ be s.t. $\|A\| = |\langle Ay, y \rangle|$

Choose closest $x \in N_\varepsilon$ $\|x-y\|_2 \leq \varepsilon$

$$|\langle Ax, x \rangle - \langle Ay, y \rangle| = |\langle Ax, x-y \rangle - \langle A(x-y), y \rangle|$$
$$\leq \|A\| \|x\| \|x-y\| + \|A\| \|x-y\| \|y\|$$
$$\leq 2\varepsilon \|A\|$$

So $\cancel{\|A\| \geq |\langle Ax, x \rangle| - 2\varepsilon\|A\|} = \cancel{(1-2\varepsilon)\|A\|}$

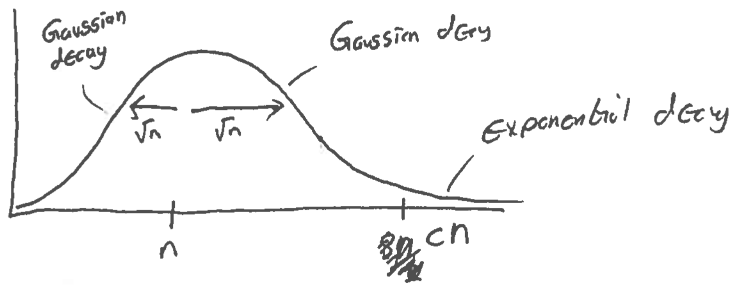
$$|\langle Ax, x \rangle| \geq |\langle Ay, y \rangle| - 2\varepsilon\|A\| = (1-2\varepsilon)\|A\|$$

and $\|A\| \leq \dfrac{|\langle Ax, x \rangle|}{1-2\varepsilon}$.

Take max over $x \in N_\varepsilon$

$$\|A\| \leq \max_{x \in N_\varepsilon} \frac{|\langle Ax, x \rangle|}{1-2\varepsilon}$$

Chi-Squared Random Variable Deviations



Roughly $\chi^2_n \approx n \pm \sqrt{n}$

Precisely:

$$P(\chi^2_n - n \le -2\sqrt{n}\, t) \le e^{-t^2}$$

$$P(\chi^2_n - n \ge \underbrace{2\sqrt{n}\, t}_{\text{bigger for small } t\text{, like Gaussian decay}} + \underbrace{2t^2}_{\text{bigger for large } t\text{, like Exponential decay}}) \le e^{-t^2}$$

changeover point
$t \approx \sqrt{n}$
or a deviation on order of n

Source: Laurent + Massart

Theorem: Let A be $N \times n$ matrix w/ iid $N(0,1)$ entries.
$\forall\, t \geq 0$
$$\sqrt{N} - C_1\sqrt{n} - t \leq \sigma_{min}(A) \leq \sigma_{max}(A) \leq \sqrt{N} + C_1\sqrt{n} + t$$
w/ prob at least $1 - 2e^{-ct^2}$.

Here C & c are constants.

Note: Could take $C_1 = 1$ for Gaussian case, but we'll prove for a larger C_1.

Proof: (Gist)

~~Easier~~ Want to show $\left\|\frac{1}{N}A^tA - I\right\|$ small

~~show~~ Find a ε Net $|N_{1/4}| \leq 9^n$

For each $X_i \in N$, show $\langle(\frac{1}{N}A^tA - I)x, x\rangle$ small whp

Add up all probabilities of failure w/ union bound.

Proof: Suffices to show

$$\left\| \frac{1}{N} A^t A - I \right\| \leq \max(\delta, \delta^2) \qquad w/\ \delta = C_1 \sqrt{\frac{n}{N}} + \frac{t}{\sqrt{N}}$$

Let $\varepsilon = \max(\delta, \delta^2)$. Note $\min(\varepsilon, \varepsilon^2) = \delta^2$.

Step 1: Approximation by a net

Note
$$\left\| \frac{1}{N} A^t A - I \right\| \leq 2 \max_{x \in \mathcal{N}} \left| \left\langle \left(\frac{1}{N} A^t A - I\right) x, x \right\rangle \right|$$
$$= 2 \max_{x \in \mathcal{N}} \left| \frac{1}{N} \|Ax\|_2^2 - 1 \right|$$

Suffices to show
$$\max_{x \in \mathcal{N}} \left| \frac{1}{N} \|Ax\|_2^2 - 1 \right| \leq \varepsilon/2.$$

Step 2: concentration for fixed item in net

Note $\|Ax\|_2^2 = \sum_{i=1}^{N} (A_i \cdot x)^2$ where A_i is row of A.

Note $A_i \cdot x \sim N(0, \|x\|_2^2) = N(0, 1)$

So $\|Ax\|_2^2 \sim \chi_N^2$

Know $P(|\chi_N^2 - N| \geq 2\sqrt{N} t + 2t^2) \leq 2e^{-t^2}$

~~Choose $t = \min(\varepsilon, \sqrt{\varepsilon}) \frac{\sqrt{n}}{8}$~~

$\cancel{P(|\chi_N^2 - N| \geq \varepsilon/2}$

$$P\left(\left|\frac{1}{N}\chi_N^2 - 1\right| \geq 2\frac{t}{\sqrt{N}} + \frac{2t^2}{N}\right) \leq 2e^{-t^2}$$

Choose $t = \min(\varepsilon, \sqrt{\varepsilon}) \frac{\sqrt{n}}{8}$

$$P\left(\left|\frac{1}{N}\chi_N^2 - 1\right| \geq \varepsilon/2\right) \leq 2e^{-\min(\varepsilon^2, \varepsilon) N/64}$$
$$\leq 2e^{-\delta^2 N/64} \leq 2e^{-\frac{C_1^2 n}{64} - \frac{t^2}{64}}$$

Step 3: Union bound

$$P\left(\max_{x\in N}\left|\frac{1}{N}\|Ax\|_2^2-1\right|\geq\frac{\varepsilon}{2}\right)\leq 9^n\, 2e^{-\frac{c_1^2 n}{64}-t^2/64}$$

$$\leq 2e^{+n\log 9-\frac{c_1^2 n}{64}-t^2/64}$$

$$\leq 2e^{-t^2/64} \quad \text{if } \frac{c_1^2}{64}>\log 9.$$

So $\left\|\frac{1}{N}A^tA-I\right\|\leq \max(\delta,\delta^2)$ w/ prob at least $1-2e^{-t^2/64}$ ∎