Activity:

Suppose $a_i \sim N(0, I_n)$ $i = 1 \dots m$

Build a vector $y = \sum_{i=1}^{m} \lambda_i a_i$ so that first component of y is larger than other components.

How big is each component?

Recovery by inexact dual certificate

Let $X_0 = e_1$ $\quad X_0 = e_1 e_1^t$ $\quad b = A X_0$

Find X such that $X \succeq 0$, $AX = b$ $\quad (*)$

Lemma: Suppose that $\exists\, Y = A^* \lambda$ such that

$\| Y_T \|_F \leq 1/2$

$Y_{T^\perp} \preceq -I_{T^\perp}$

$\Big\}$ weakening of dual certificate

and $\forall$ feasible $X_0 + H$, $\quad \| H_{T^\perp} \|_* \geq \dfrac{0.94\,(1-1/8)}{1+1/8} \| H_T \|$

$\Big\}$ strengthening of injectivity

Then X_0 is unique soln to $(*)$

Proof: Let $X_0 + H$ be feasible $\quad AH = 0$

$0 = \langle \lambda, AH \rangle = \langle A^* \lambda, H \rangle = \langle Y, H \rangle = \langle Y_T, H_T \rangle + \langle Y_{T^\perp}, H_{T^\perp} \rangle$

So $\quad -\langle Y_{T^\perp}, H_{T^\perp} \rangle = + \langle Y_T, H_T \rangle$

$\| H_{T^\perp} \|_* \leq \| Y_T \|_F \| H_T \|_F \leq \frac{1}{2} \| H_T \|_F \leq \frac{1}{\sqrt{2}} \| H_T \|$

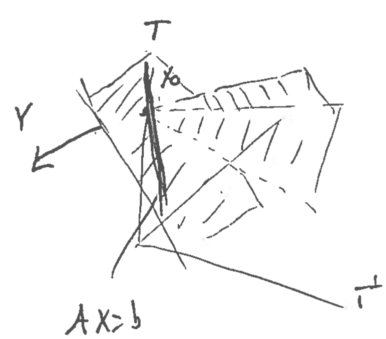
We have $\quad \| H_{T^\perp} \|_* \leq \frac{1}{\sqrt{2}} \| H_T \|$

$\| H_{T^\perp} \|_* \geq \dfrac{0.94\,(1-1/8)}{1+1/8} \| H_T \|$

$\Rightarrow \left(\dfrac{0.94\,(1-1/8)}{1+1/8} - \dfrac{1}{\sqrt{2}} \right) \| H_T \| \leq 0$

$\Rightarrow H_T = 0 \Rightarrow H_{T^\perp} = 0$ $\quad \blacksquare$

Picture



Dual certificate defines a hyperplane that separates $AX = b$ from cone

Conic condition for feasible matrices & RIP-1 bound

Lemma: Suppose $\frac{1}{m}\|AX\|_1 \leqslant (1+\frac{1}{8})\ tr(X) \qquad \forall\ X \succcurlyeq 0$

and $\frac{1}{m}\|AX\|_1 \geqslant 0.94(1-\frac{1}{8})\|X\| \qquad \forall$ sym rank-2 X.

Then if $X_0 + H$ is feasible,

$$\|H_{T^\perp}\|_* \geq \frac{0.94(1-\frac{1}{8})}{1+\frac{1}{8}}\|H_T\|$$

Proof: $\overset{A(X_0+H)=b}{\Rightarrow} 0 = AH = AH_T + AH_{T^\perp}$,

So $\frac{1}{m}\|AH_T\|_1 = \frac{1}{m}\|AH_{T^\perp}\|_1$

$0.94(1-\frac{1}{8})\|H_T\| \leqslant (1+\frac{1}{8})\|H_{T^\perp}\|_*$

as H_T is rank 2 as $H_{T^\perp} \succcurlyeq 0$.

Recovery Under a Gaussian Model

Let $a_i \sim N(0, I_n)$ $i = 1 \dots m$. $AX = \{\langle X, a_i a_i^t \rangle\}_{i=1\dots m}$

To show $X_0 X_0^t$ is unique element of $\{X \succcurlyeq 0,\ AX = A x_0 x_0^t\}$

with high probability, we will show

- $\frac{1}{m} \|AX\|_1 \leq (1+\frac{1}{8})\ \mathrm{tr}(X) \quad \forall\ X \succcurlyeq 0$
- $\frac{1}{m} \|AX\|_1 \geq 0.94(1-\frac{1}{8}) \|X\| \quad \forall$ sym rank-2 X.

RIP-like property

- $\exists\ Y = A^* \lambda$ such that
  - $\|Y_{T^\perp} + \frac{17}{10} I_{T^\perp}\| \leq \frac{1}{10}$
  - $\|Y_T\|_F \leq \frac{3}{20}$

Existence of dual cert

Lemma:

Let $a_i \sim N(0, I_n)$

Let $AX = \{\langle X, a_i a_i^t \rangle\}_{i=1\dots m} = \{a_i^t X a_i^t\}_{i=1\dots m}$

If $m \geq C_\delta n$ then $\forall X$ $\frac{1}{m}\|AX\|_1 \leq (1+\delta)\|X\|_*$ w/ prob $1-2e^{-m\gamma_\delta}$

Proof: Suffices to prove for $X = vv^t$ w/ $\|v\|=1$.

Because: Let $X = \sum_{i=1}^n \lambda_i v_i v_i^t$. $\frac{1}{m}\|AX\|_1 \leq \sum_{i=1}^n |\lambda_i| (1+\delta) 1 = \|X\|_*$.

$$\|A vv^t\|_1 = \sum_i |a_i \cdot v|^2 = \|Zv\|^2 \quad \text{w/} \quad Z = \begin{pmatrix} -a_1- \\ -a_2- \\ \vdots \\ -a_m- \end{pmatrix}$$

$$\frac{1}{m}\|A vv^t\|_1 \leq \sigma_{max}^2\left(\frac{Z}{\sqrt{m}}\right).$$

Concentration estimate: $P(\sigma_{max}(Z) > \sqrt{m} + \sqrt{n} + t) \leq e^{-t^2}$

Choose $m \geq \frac{n}{(\delta/2)^2}$ $t = \sqrt{m}\,\delta/2$

$$\Rightarrow P\left(\sigma_{max}\left(\frac{Z}{\sqrt{m}}\right) > 1+\delta\right) \leq e^{-\delta^2 m/2}.$$