Exercise: Consider $\mathbb{R}^2$

What is $\partial \|\cdot\|_2(x)$?

$$\begin{cases} \{ z \in \mathbb{R}^2; \ \|z\|_2 \leq 1 \} & \text{if } x = 0 \\ \dfrac{x}{\|x\|} & \text{if } x \neq 0 \end{cases}$$

Derivation of Dual Problem to basis pursuit

$$\min \|x\|_1 \quad \text{st} \quad Ax = b$$

$$\mathcal{L}(x,\lambda) = \|x\|_1 - \langle \lambda, Ax - b\rangle$$

$$g(\lambda) = \inf_{x} \mathcal{L}(x,\lambda) = \inf \|x\|_1 - \langle A^t\lambda, x\rangle + \langle \lambda, b\rangle$$

$$= \langle \lambda, b\rangle + \inf_{x} \|x\|_1 - \langle A^t\lambda, x\rangle$$

Study $\inf_x \|x\|_1 - \langle A^t\lambda, x\rangle$. When is it $-\infty$? when finite.

Just choose x that selects largest index of $A^t\lambda$

If $\|A^t\lambda\|_\infty > 1$, $\inf = -\infty$.

What does it take to minimize $\|x\|_1 - \langle A^t\lambda, x\rangle = f(x)$

$$0 \in \partial f(x) \Rightarrow 0 \in \partial\|x\|_1 - A^t\lambda. \Rightarrow A^t\lambda \in \partial\|x\|_1(x)$$

So $(A^t\lambda)_i = \begin{cases} \text{sign } x_i & \text{if } x_i \neq 0 \\ \in [-1,1] & \text{if } x_i = 0 \end{cases}$ possible if $\|A^t\lambda\|_\infty \leq 1$

~~Choose $x = \text{sign}(A^t\lambda)$. So $\|x\|_1 - \langle A^t\lambda, x\rangle = \|x\|_1 - \langle A^t\lambda, \text{sign } A^t\lambda\rangle = \|\text{sign } A^t\lambda\|_1$~~

$$\|x\|_1 - \langle A^t\lambda, x\rangle = \|x\|_1 - \langle \text{sign } x, x\rangle = \|x\|_1 - \|x\|_1 = 0.$$

So $\inf_x \|x\|_1 - \langle A^t\lambda, x\rangle = \begin{cases} 0 & \text{if } \cancel{(A^t\lambda)_i = \text{sign } x_i \text{ if } x_i \neq 0} \ \|A^t\lambda\|_\infty \leq 1 \\ -\infty & \text{otherwise} \end{cases}$

Dual problem

$$\sup_\lambda g(\lambda) = \sup \langle \lambda, b\rangle \quad \text{st} \quad \|A^t\lambda\|_\infty \leq \cancel{\text{sum}}\ 1$$

By strong duality of LPs, $\sup_\lambda g(\lambda) = \inf \|x\|_1$ st $Ax = b$