Convex functions

Let $f: \mathbb{R}^n \to \mathbb{R}$

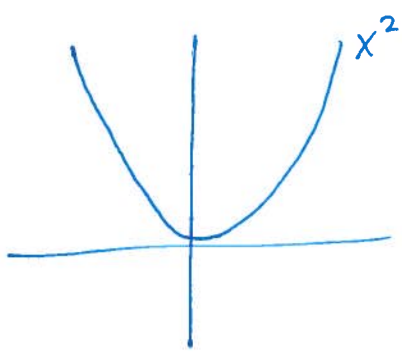
We say f is convex if $f(\alpha x + (1-\alpha) y) \le \alpha f(x) + (1-\alpha) f(y) \quad \forall x, y$
$\alpha \in [0,1]$

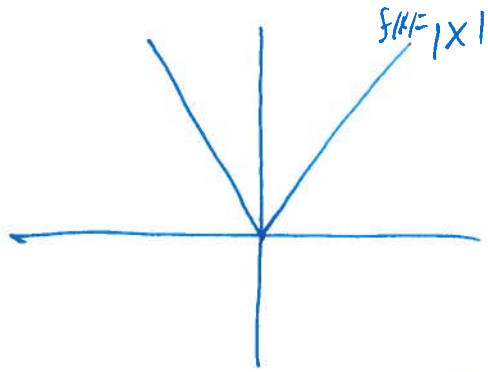
"f lives below its secant lines"
"f 'curves' up" (even if no 2^{nd} deriv)

"function of an average is less than average of function)

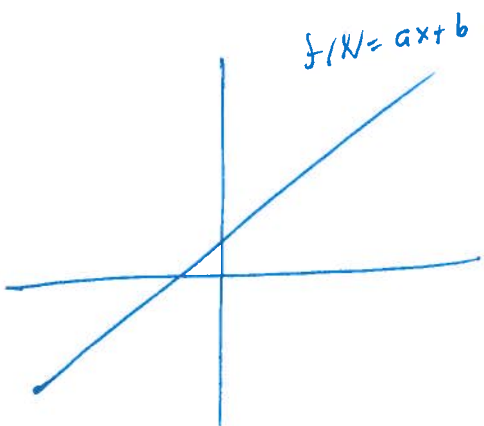
The term $\alpha x + (1-\alpha) y$ for $\alpha \in [0,1]$ is a "convex combination" of x & y

Eg:

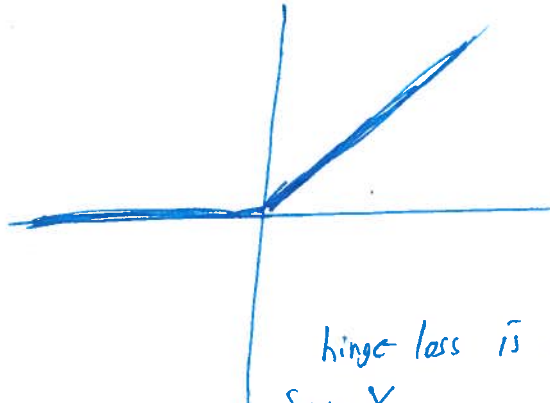




convex functions may be <u>nonsmooth</u>



linear functions are convex



hinge loss is convex
$f(x) = X_+$

High dimensional convex functions. Let $x \in \mathbb{R}^n$

$\|x\|_2^2$

$\|x\|_2$

$a \cdot x + b$ for $a \in \mathbb{R}^n$, $b \in \mathbb{R}$

$\|x\|_1$

$\|x\|_\infty$

$\|x\|$ for any norm

$f(x,y) = \frac{x^2}{y}$ is convex on $\mathbb{R} \times \mathbb{R}_{++}$

Properties

- weighted sum of convex function is convex (if weights nonnegative)
- max of convex functions is convex
- composition of convex function w/ affine function is convex

- If f is twice diff'able

  f is convex in Ω iff $\nabla^2 f(x) \succeq 0$ $\forall x \in \Omega$.

Activity: Convex or not

$$f(x,y) = xy \quad \text{on} \quad \{x>0\} \times \{y>0\}$$

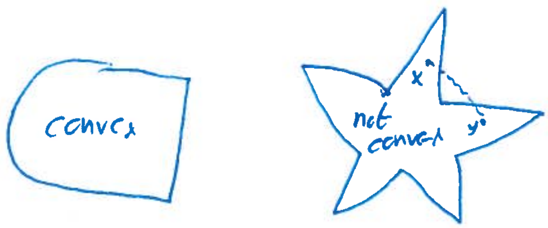
$$f(x) = \|a \cdot x - b\|_2^2$$

$$f(x) = ((a \cdot x)^2 - b^2)^2$$

Convex Sets

Let $S \subset \mathbb{R}^n$

S is <u>convex</u> if $x, y \in S \Rightarrow \alpha x + (1-\alpha) y \in S$.



"the line between any two points is in the set"

"set always bulges out"

Intersection of convex sets is convex.

Convex or not (Draw set)

$\{(x,y)\in\mathbb{R}^2 \mid xy \geq 1\}$

$$\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \begin{array}{l} xyz \geq 1 \\ x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{array}\right\}$$

Fix $a\in\mathbb{R}^n$

$\cancel{\{x \mid \frac{|\langle a_i, x\rangle|}{\|x\|_2} \geq 1/2\}}$

$\{x\in\mathbb{R}^n \mid \langle a_i, x\rangle \geq \frac{1}{2}\|x\|_2\}$

$\{X\in\mathbb{R}^{n\times n},\ X \succeq 0\}$