29 September 2016
Analysis I
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Week 7 — Summary — Completion of Vector Spaces & Open and Closed sets

66. A relation, $\sim$, on a set X is an equivalence relation if it is reflexive, symmetric, and transitive. That is, if for all $a, b, c \in X$
    (a) $a \sim a$ (reflexivity)
    (b) $a \sim b \Rightarrow b \sim a$ (symmetry)
    (c) $a \sim b$ and $b \sim c \Rightarrow a \sim c$ (transitivity)
67. Given a set X and an equivalence relation $\sim$, the equivalence class of an element $a \in X$ is the set of elements equivalent to a. The set of equivalence classes is denoted by $X \;/ \sim$. We can define operations (e.g. addition, multiplication) on equivalence classes if the operation is well defined (is independent of which representative is chosen from the equivalence classes).
68. We can define an equivalence relation between two Cauchy sequences of a (not necessarily complete) normed vector space:
    $$\{x_n\} \sim \{y_n\} \text{ if and only if } \lim_{n\to\infty}(x_n - y_n) = 0$$
    The set of equivalence classes forms a normed vector space.
69. $\mathbb{R}$ can be defined as the set of equivalence classes of Cauchy sequences of $\mathbb{Q}$. This is called the completion of $\mathbb{Q}$.
70. The completion of a normed vector space is defined as the set of equivalence classes of Cauchy sequences of elements in the space. The completion is a complete normed vector space.

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71. Definition: A subset S of a normed vector space is open if for any $x \in S$, there is an open ball (centered at x) contained within S.
72. Definition: A subset S of a normed vector space is closed if its complement is open.
73. The finite intersection of open sets is open.
74. The arbitrary union of open sets is open.
75. The finite union of closed sets is closed.
76. The arbitrary intersection of closed sets is closed.
77. Definition: A point x is a limit point of a set S if there are points in S that are arbitrarily close to x under the provided norm.
78. A set is closed if and only if it contains all its limit points.
79. Definition: The closure of a set is the collection of limit points of that set. Write the closure of S as $\bar{S}$.
80. The closure of a set S is the intersection of all closed sets containing S.

81. Definition: Let $S \subset T$. The set S is dense in the set T if $T \subset \bar{S}$.

82. A function f from one normed vector space to another is continuous if $\lim_{x \to a} f(x) = f(a)$. That is, if $\forall \varepsilon,\ \exists \delta$ such that $\|x - a\| \leq \delta \Rightarrow \|f(x) - f(a)\| < \varepsilon$.

83. A function is continuous if and only if the inverse image of any open set is open.