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Analysis I
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Pledged HW 6

Time limit: 3 hours. You may not use your books, your homeworks, your notes, or any electronics during this timed homework. Please write the start and finish times on your paper. To receive full credit, you must name all major theorems and state definitions used in your arguments. All counter examples must be accompanied by a proof. You may cite results from class and well-known theorems.

This homework is pledged. On the first page, please write your signature and the Rice University pledge: "On my honor, I have neither given nor received any unauthorized aid on this homework."

Due: Thursday, 19 October 2017 at the beginning of class.

1. (10 points) Prove that $x^3 - 2x^6 \leq \log(1 + x^3) \leq x^3 + 2x^6$ for all $|x| \leq 1/2$.
   Note: log is the logarithm base e.

2. (a) (5 points) Find a continuous function $g : (0, 1) \to \mathbb{R}$ that is not uniformly continuous. Prove it.
   (b) (10 points) Prove or disprove: If $f_n : (0, 1) \to \mathbb{R}$ are continuous and $f_n \to f$ uniformly for some $f : (0, 1) \to \mathbb{R}$, then f is uniformly continuous.
   (c) (10 points) Prove or disprove: If $f_n : (0, 1) \to \mathbb{R}$ are uniformly continuous and $f_n \to f$ uniformly for some $f : (0, 1) \to \mathbb{R}$, then f is uniformly continuous.

3. (15 points) Find a sequence of $C^1([0, 1])$ functions that is Cauchy with respect to the sup norm but is not Cauchy with respect to the C^1 norm. Recall that the C^1 norm of a function is $\|f\|_{C^1} = \sup_x |f(x)| + |f'(x)|$.

4. (a) (10 points) Prove that if f and g are Riemann-integrable real-valued functions on $[0, 1]$, then $h(\cdot) = f(\cdot)g(\cdot)$ is Riemann integrable.
   (b) (10 points) Prove or disprove: if f and g are Riemann-integrable real-valued functions on $[0, 1]$ and if $f(x)$ and $g(x)$ are nonzero for all $x \in [0, 1]$, then $h(\cdot) = f(\cdot)/g(\cdot)$ is Riemann integrable.

1) Prove: $x^3 - 2x^6 \leq \log(1+x^3) \leq x^3 + 2x^6$ for $|x| \leq \frac{1}{2}$

By Taylor Remainder theorem, for $f \in C^2$

$$\left| f(y) - (f(0) + f'(0)y) \right| \leq \frac{C\,y^2}{2} \quad \text{on } |y| \leq a,$$

where $C = \max\limits_{|y| \leq a} |f''(y)|$

Let $f(y) = \log(1+y)$. So $f(0) = 0$ & $f'(0) = 1$.
Let $a = \frac{1}{2}$. So $C = \max\limits_{|y| \leq a} \left| \frac{-1}{(1+y)^2} \right| = \frac{1}{(\frac{1}{2})^2} = 4$

So $\left| \log(1+y) - y \right| \leq 2y^2$ on $|y| \leq \frac{1}{2}$.

We conclude by letting $y = x^3$.

2a) Let $g(x) = \frac{1}{x}$.

g is not uniformly continuous.

We will prove:

$\forall \varepsilon \quad \forall \delta \quad \exists x, y$ st $|x-y| < \delta$ & $|g(x) - g(y)| > \varepsilon$

Fix ε. Fix $\delta < \frac{1}{\varepsilon}$. Let $x = \delta$, $y = \delta/2$.

Observe $|g(x) - g(y)| = \left|\frac{1}{\delta} - \frac{2}{\delta}\right| = \left|\frac{-1}{\delta}\right| = \frac{1}{\delta} > \varepsilon$ ∎

b) False.

Let $f_n : (0,1) \to \mathbb{R}$

$x \mapsto \frac{1}{x}$.

$f_n \to g$ uniformly as $\|f_n - g\|_\infty = 0 \quad \forall n$.

c)

As $f_n \to f$ uniformly,

$\forall \varepsilon \; \exists N$ st $\quad n \geq N \Rightarrow \sup_x |f_n(x) - f(x)| < \varepsilon/3 \qquad (*)$

As f_N is uniformly continuous,

$\forall \varepsilon \; \exists \delta_N$ st $\quad |x-y| < \delta_N \Rightarrow |f_N(x) - f_N(y)| < \varepsilon/3 \qquad (**)$

Fix ε. Let N be given by $*$. Let δ_N be given by $(**)$

If $|x-y| < \delta_N$, then

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)|$$

$$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3$$

$$\leq \varepsilon$$

3) Let $f_n(x) = \dfrac{\sin(n^2 x)}{n}$.

The seq $\{f_n\}$ is Cauchy wrt sup norm as $f_n \to 0$ wrt. sup norm.

If $\{f_n\}$ is Cauchy wrt C^1 norm, $\|f_n\|_{C_1}$ must be bounded. But $\|f_n\|_{C^1} = \sup_x |n \sin(n^2 x)| + \left|\dfrac{\sin(n^2 x)}{n}\right|$

$\geq n - \frac{1}{n} \to \infty$ as $n \to \infty$.

4a)

We may write $f(x)g(x) = \dfrac{(f(x)+g(x))^2 - (f(x)-g(x))^2}{4}$

Thus it suffices to show that

- $f \pm g$ is Riemann integrable
- if f is Riemann integrable, then f^2 is Riemann integrable

Claim: If $f:[0,1] \to \mathbb{R}$ is R Int, and $g:[0,1]\to\mathbb{R}$ is too, then $f+g$ & $f-g$ are R int.

Proof Fix ε. Let P_f be a partition st

$$|M(f,P_f) - m(f,P_f)| \leq \varepsilon$$

Similarly for P_g.

$$|M(g,P_g) - m(g,P_g)| \leq \varepsilon.$$

Let P be joint refinement of P_f & P_g.

On P we have

$$|M(f,P) - m(f,P)| \leq \varepsilon$$

& $|M(g,P) - m(g,P)| \leq \varepsilon.$

As $M(f+g,P) \leq M(f,P) + M(g,P)$

and $m(f+g,P) \geq m(f,P) + m(g,P)$

we have

$$|M(f+g,P) - m(f+g,P)| \leq 2\varepsilon \;\blacksquare$$

As $M(f-g,P) \leq M(f,P) - m(g,P)$

& $m(f-g),P) \geq m(f,P) - M(g,P)$

we have

$$|M(f-g,P) - m(f-g,P)| \leq 2\varepsilon \;\blacksquare$$

Claim:

If $f:[0,1]\to\mathbb{R}$ is Riemann integrable, then f^2 is Riemann integrable

Proof:

As f is R. Integrable, $|f(x)|\leq\hat{M}$ for some $\hat{M}$.

Note: for any x,y $|f^2(x)-f^2(y)| = |f(x)+f(y)|\,|f(x)-f(y)|$
$$\leq 2\hat{M}\,|f(x)-f(y)|$$

Fix ε. Let P be such that
$$M(f,P)-m(f,P)\leq \varepsilon/2\hat{M}$$

Then
$$\sup_{x\in[x_i,x_{i+1}]} f^2(x) - \inf_{x\in[x_i,x_{i+1}]} f^2(x) = \sup_{x,y\in[x_i,x_{i+1}]} f^2(x)-f^2(y)$$
$$\leq 2\hat{M}\sup_{x,y\in[x_i,x_{i+1}]}|f(x)-f(y)|$$

So
$$M(f^2,P)-m(f^2,P)\leq\sum_{i=0}^{n-1}2\hat{M}\sup_{x,y\in[x_i,x_{i+1}]}|f(x)-f(y)|\,(x_{i+1}-x_i)$$
$$=2\hat{M}\,(M(f,P)-m(f,P))$$
$$\leq 2\hat{M}\,\frac{\varepsilon}{2\hat{M}}=\varepsilon\ \blacksquare$$

4b) Let $f(x) = 1 \quad \forall x$

Let $g(x) = \begin{cases} 1 & \text{if } x = 0 \\ x & \text{if } x \neq 0. \end{cases}$

g is Riemann integrable as it is the sum of $\begin{cases} 1 & \text{if } x=0 \\ 0 & \text{o'wise} \end{cases}$ & $x \mapsto x$, which are both Riemann integrable.

$\frac{f}{g}$ is unbounded, thus the upper Riemann sum of $\frac{f}{g}$ is undefined. Hence $\frac{f}{g}$ is not Riemann integrable.