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Analysis I
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Day 11 — Summary — Inner Products and Equivalent Norms

57. An inner product $\langle \cdot, \cdot \rangle$ satisfies the following axioms for all $u, v, w \in V$:
    (a) $\langle v, w \rangle = \langle w, v \rangle$
    (b) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
    (c) If $c \in \mathbb{R}$, $\langle cv, w \rangle = c\langle v, w \rangle = \langle v, cw \rangle$
    (d) $\langle v, v \rangle \geq 0 \; \forall v$ and $\langle v, v \rangle = 0 \Rightarrow v = 0$.
58. Inner products induce a norm $\|v\| = \sqrt{\langle v, v \rangle}$.
59. Inner products satisfy the Cauchy-Schwarz inequality $\langle v, w \rangle \leq \|v\|\|w\|$.
60. Definition: Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent on a vector space V if there exists $c, C > 0$ such that
$$c\|x\|_b \leq \|x\|_a \leq C\|x\|_b \; \forall x \in V.$$
61. All norms on finite dimensional vectors spaces, e.g. $\mathbb{R}^n$, are equivalent.
62. In infinite dimensional vector spaces, some pairs of norms are not equivalent.

58) $\|v\| = \sqrt{\langle v, v \rangle}$ is a norm

Triangle inequality

$$\|v+w\| = \sqrt{\langle v+w, v+w \rangle}$$
$$= \sqrt{\langle v,v \rangle + 2\langle v, w \rangle + \langle w, w \rangle}$$
$$\leq \sqrt{\|v\|^2 + 2\|v\|\,\|w\| + \|w\|^2}$$
$$= \sqrt{(\|v\| + \|w\|)^2}$$
$$= \|v\| + \|w\| \;\blacksquare$$

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Cauchy Schwarz

$$\langle v, w \rangle \leq \|v\| \, \|w\|$$

Compare to: dot product angle formula

$x, y \in \mathbb{R}^n$ $\quad x \cdot y = |x|\,|y| \cos\theta$ $\quad$ so $|x \cdot y| \leq |x|\,|y|$.

Proof: WLOG let $\|v\| = 1$, $\|w\| = 1$.

We show $\langle v, w \rangle \leq 1$.

Consider $0 \leq \langle v-w, v-w \rangle = \|v\|^2 - 2\langle v, w \rangle + \|w\|^2$

$= 2 - 2\langle v, w \rangle$

So $2\langle v, w \rangle \leq 2 \Rightarrow \langle v, w \rangle \leq 1$

Repeat w/ $0 \leq \langle v+w, v-w \rangle$ to get $\langle v, w \rangle \geq -1$

When is inequality ~~strict~~ achieved?

When $\frac{v}{\|v\|} = \pm \frac{w}{\|w\|}$. That is, when $v = cw$

Generalizations: ~~In~~ For functions and sequences

$$\langle v, w \rangle \leq \|v\|_1 \, \|w\|_\infty$$

$$\langle v, w \rangle \leq \|v\|_p \, \|w\|_q \qquad \tfrac{1}{p} + \tfrac{1}{q} = 1 \quad \text{(Hölder)}$$

60) Equivalent norms

Background: Norms are used to define a notion of convergence, completeness, open sets, etc.

Some pairs of norms will produce same notion of convergence, completeness; some will produce different notions.

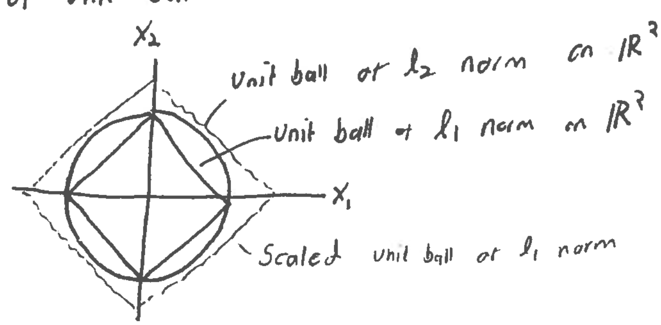
Eg a seq may converge in one norm but not another

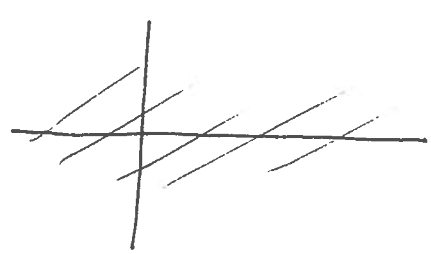
Idea: Two norms are equivalent when all norm-defined notions are the same.

Eg a seq x_n conv in one norm $\Rightarrow$ seq x_n converges in other norm

Dfn: $\|\cdot\|_a$ Equivalent to $\|\cdot\|_b$ if $\exists\ c, C$ st $c\|x\|_b \le \|x\|_a \le C\|x\|_b$

Visualization: Unit ball of each norm is contained in some multiple of unit ball of the other





61) Example: In $\mathbb{R}^n$, $\|\cdot\|_1$ equivalent to $\|\cdot\|_\infty$

Proof: ~~Need to show~~ $\cancel{c\|X\|_\infty \leq \|X\|_1 \leq C\|X\|_\infty}$

We will show $\|X\|_\infty \leq \|X\|_1 \leq n\,\|X\|_\infty$

Left inequality: immediate

Right inequality: $\|X\|_1 = \sum_{i=1}^{n} |X_i| \leq \sum_{i=1}^{n} \max |X_i| = n \max |X_i|$

$= n\,\|X\|_\infty.$

62 ~~B~~) Example: Consider vector space ℓ_1

($\text{sequences } x_i \in \mathbb{R} \text{ st } \sum_{i=1}^{\infty} |x_i| < \infty$).

The ℓ_1 norm and ℓ_∞ norm are not equivalent.

Pf: ~~Suffices to exhibit a sequence of~~

Suffices to exhibit for any N an X such that $\|X\|_\infty = 1$ $\|X\|_1 = N$

Fix N. Let $X_i = \begin{cases} 1 & i \le N \\ 0 & i > N \end{cases}$. $\|X\|_\infty = 1$ $\|X\|_1 = N$ ∎

Exercise: ~~Find a seq~~ $\cancel{\|X\|_1 = 1}$ $\cancel{\|X\|}$

Exhibit a collection of X showing that ℓ_1 and ℓ_2 norm are not equivalent.

Exercise: Find C such that $\|X\|_1 \leqslant C\|X\|_2$ for $X \in \mathbb{R}^n$

— C_1 such that $\|X\|_2 \leqslant C_1'\|X\|_1$

Activity:

In $\mathbb{R}^n_1$ prove $\|x\|_1 \leq \|x\|_2$

In ℓ^1 prove $\|x\|_1 \leq \|x\|_2$

Activity

In C^1: Is $f \mapsto \|f\|_\infty$ equivalent to $f \mapsto \|f\|_\infty + \|f'\|_\infty$?

In ℓ^1: is $x \mapsto \|x\|_1 + \|x\|_2$ equivalent to $x \mapsto \|x\|_1$?

In ℓ^1: is $x \mapsto \|x\|_1 + \|x\|_2$ equivalent to $x \mapsto \|x\|_2$?