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Analysis I
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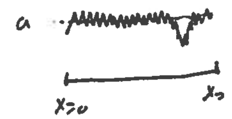
Day 5 — Summary — Differentiability, Mean Value Theorem

1. The derivative of f at x is $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, if this limit exists. A function is differentiable on a set if it is differentiable at every point in that set.
2. Product rule, quotient rule, chain rule.
3. Differentiability implies continuity.
4. Let $C^p([a,b])$ be the set of functions defined on $[a,b]$ that are differentiable p times, and the p-th derivative is continuous. Let C^∞ be the set of functions that are in C^p for all p.
5. At a local maximum (or minimum) of a differentiable function, the derivative is zero (provided that this max or min occurs in the interior of the function's domain).
6. Mean value theorem: If f is continuous on $[a,b]$ and is differentiable on (a,b), then for some $c \in (a,b)$, $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Warm up



a) $f \in C(0,1)$

Can f cross value a ∞ly many times?

yes $f(x) = \sin\frac{1}{x}$ crosses 0 ∞ly many times

b) $f \in C[0,1]$

Can f cross value a ∞ly many times?

yes $f(x) = x \sin\frac{1}{x}$ crosses 0 ∞ly many times

c) $f \in C(0,1)$

Can f cross $[-\varepsilon, \varepsilon]$ an ∞ # times for a fixed $\varepsilon > 0$?

yes $f(x) = \sin\frac{1}{x}$ crosses $(-\frac{1}{2}, \frac{1}{2})$ ∞ly many times

d) $f \in C[0,1]$

Same question

no. Take points where $f(x) = a$. By Bolzano Weierstrass, a subseq converges to some $x^* \in [0,1]$. By continuity, $f(x^*) = a$, yet there are arbitrarily close x that adopt value ε.

1) Differentiability examples:

Continuous on $[0,1]$
Diff'able on $(0,1)$ but not $[0,1]$:

$$\sqrt{1-x^2} \quad , \quad \sqrt{x}$$

Defined on $\mathbb{R}$ but diff'able no where

$$1_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{o'wise} \end{cases}$$

Defined on $\mathbb{R}$ and diff'able only at $x=0$

$$x^2 1_{\mathbb{Q}}(x)$$

4) Examples

a) $f \in C^0(\mathbb{R})$
$f \notin C^1(\mathbb{R})$:

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x & x \geq 0 \end{cases}$$

b) $f \in C^p(\mathbb{R})$
$f \notin C^{p+1}(\mathbb{R})$:

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x^p & x \geq 0 \end{cases}$$

c) $f \in C^\infty(\mathbb{R})$
f is bounded :
f is not constant

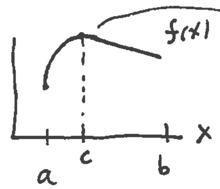
$f(x) = \sin x$
or
e^{-x^2}

~~d)~~ $\cancel{f \in C^1(0,1)}$:
$\cancel{f \notin C^1[0,1]}$

5) If f has local max at interior point x, and f diff'able, $f'(x) = 0$.

Formally: If f diffable on (a,b) and $f(c) \geq f(x)$ $\forall x \in (a,b)$ then $f'(c) = 0$.

Visually:



at max, deriv is 0.

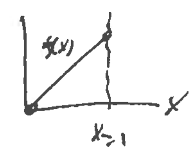
Application: Find optimizers by setting derivative to 0.

Proof:
(Sketch)

$$\frac{f(x+h)-f(x)}{h}$$ is
- ≥ 0 ~~nonnegative~~ for negative h
- ≤ 0 ~~nonpositive~~ for positive h.

Hence limit must be 0

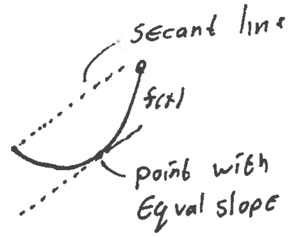
NonExample if max occurs at boundry: $f(x) = x$ on $[0,1]$



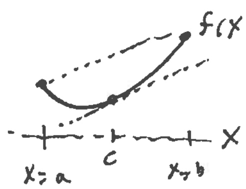
6) Mean value theorem:

Idea: For any secant line, theres a point with same slope

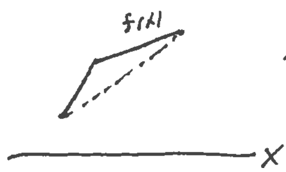
Visual:



Precisely: If f continuous on $[a,b]$, diffable on (a,b), $\exists\, c$ st $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

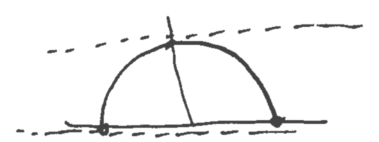


Nonexample:

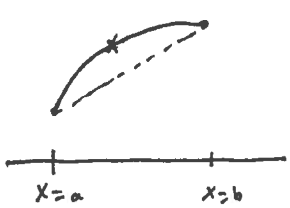


- not diffable at one point.
no where has slope of secant line

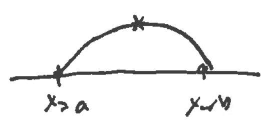
Note: f doesnt need to be diffable on $[a,b]$,
MVT applies to $\sqrt{1-x^2}$ on $[-1,1]$



Proof:



Subtract away secant line



At max or min, slope must be 0.

Application: Error bound in Taylor Serig

If f is smooth, $f(x) \approx f(0) + f'(0)x$

Let $e(x) = f(x) - f(0) - f'(0)x$. Claim: $|e(x)| \leq \max |f''| \, x^2$.

Proof:
$$
\begin{aligned}
e(x) &= e(x) - e(0) \\
&= e'(c_1)\, x && \text{by MVT} \\
&= [f'(c_1) - f'(0)]\, x \\
&= f''(c_2)\, c_1 \, x && \text{by MVT}
\end{aligned}
$$

$$
\begin{aligned}
|e(x)| &\leq \max |f''| \, |c_1| \, |x| \\
&\leq \max |f''| \, |x|^2 \qquad \text{as } c_1 < x.
\end{aligned}
$$