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Analysis I
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Day 24 — Summary — Almost everywhere convergence and Lebesgue Integration

1. Definition: The functions f and g are equal almost everywhere if $\mu(\{x \mid f(x) \neq g(x)\}) = 0$. For many purposes, these functions should be considered to be equal.

2. Definition: $f_n \to f$ almost everywhere if the set of points where f_n does not converge to f has measure zero.

3. If f_n is an L^1-Cauchy sequence of step maps, then there is a subsequence that converges almost everywhere. Further, for any ε, there is a set with measure less than ε outside of which the convergence is uniform.

4. Definition: A simple function is a function that adopts finitely many values: $\phi(x) = \sum_{n=1}^{N} a_n 1_{E_n}$.

5. Define the Lebesgue integral of a nonnegative simple function ϕ as $\int_{\mathbb{R}} f d\mu = \sum_{n=1}^{N} a_n \mu(E_n)$.

6. Define the Lebesgue integral of a nonnegative measurable function f as
$$\int_{\mathbb{R}} f d\mu = \sup \left\{ \int_{\mathbb{R}} \phi d\mu \;\middle|\; \phi \text{ simple, } 0 \leq \phi \leq f \right\}$$

7. Define the Lebesgue integral of a not-necessarily-nonnegative function $f = f^+ - f^-$, where f^+ and f^- are both nonnegative, as
$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu,$$
provided both integrals on the right hand side are not infinite.