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Analysis I
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Day 21 — Summary — Extension of linear operators

1. Definition: A linear operator (aka function or map) L from a normed vector space to another normed vector space is bounded if $\|L(x)\| \leq C\|x\|$ for all x. The constant C is an operator bound for L. The smallest such C is the operator norm of L.

2. A linear map from a normed vector space to another normed vector space is continuous if and only if it is bounded (as an operator).

3. Let F be a normed vector space, and let F_0 be a subspace. The closure of F_0 in F is a subspace of F.

4. Let F be a normed vector space, and let F_0 be a subspace. Let $L : F_0 \to E$ be a continuous linear map from F_0 into the complete normed vector space E. Then L has a unique extension to a continuous linear map $\bar{L} : \bar{F_0} \to E$ with the same operator bound.

3) $F_0 \subseteq F$ — normed vector space

Subspace

Then $\overline{F_0}$ is a subspace of F

Strict subspace whose closure is whole space

Example: $C^1[0,1] \subseteq (C^0[0,1], \|\cdot\|_\infty)$

$$\overline{C^1[0,1]} = C^0[0,1] \quad (\text{under } \|\cdot\|_\infty)$$

As any element of C^0 can be approximated arbitrarily well by something in C^1.

visually  approximated by 

Activity: Operator that smooths is $\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} f(y)\,dy$. Draw what it does to  & 

what is operator bound

Strict subspace whose closure is a different strict subspace

$$C^1[0,1] \subseteq (L^\infty[0,1], \|\cdot\|_\infty)$$

$$\overline{C^1[0,1]} = C^0[0,1] \subset L^\infty[0,1]$$

Use: If we define a continuous linear operator on F_0, we will be able to extend it to $\overline{F_0}$ by continuity

Proof: Easy

4) $F_0 \subseteq F$, a normal vector space

$L: F_0 \to E$ be continuous, linear map, E is complete normed vector space

L has a! extension to $\bar{L}: \bar{F}_0 \to E$

If $\|Lx\| \le c\|x\|$ then $\|\bar{L}x\| \le c\|x\|$

Example: $F = \ell^1$ w/ ℓ^1 norm

$F_0 = \{$ seq w/ finite support $\}$

$Lx = \sum_{i=0}^{N} x_i$ w/ N max nonzero coef in x is linear and continuous

$\bar{F}_0 = F$

$\bar{L} = \sum_{i=1}^{\infty}$

~~Illustration~~ Analogy: Suppose $f: \mathbb{R} \to \mathbb{R}$ and you are given f on $\mathbb{Q}$ and the fact that f is continuous

Then, $\exists\ \bar{f}: \bar{\mathbb{Q}} \to \mathbb{R}$ also continuous

Let $x_i \to x_\infty$ $f(x_\infty) = \lim_{i \to \infty} f(x_i)$