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Analysis I
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Day 20 — Summary — Power series

1. For any power series $\sum a_n x^n$, there is a radius of convergence R (which may be zero, finite, or infinite), such that the series converges absolutely for all $|x| < R$ and does not converge absolutely for any $|x| > R$.

2. The radius of convergence of $\sum a_n x^n$ is $1/\limsup_{n\to\infty} |a_n|^{1/n}$.

3. Let $f(x) = \sum a_n x^n$ be a power series with radius of convergence $R > 0$. Then, for all $|x| < R$, $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and this sum converges absolutely for all $|x| < R$.

4. Let $\{f_n\}$ be a sequence of functions in $C^1([a,b])$ and assume that $f_n' \to g$ uniformly, and that $f_n(x_0)$ converges for some x_0. Then, there exists a function f such that $f_n \to f$ uniformly, and f is differentiable, and $f' = g$.

5. Let $f(x) = \sum a_n x^n$ be a power series with radius of convergence $R > 0$. Then, an antiderivative of $f(x)$ in $-R < x < R$ is given by $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ and this sum converges absolutely for all $|x| < R$.

Warmup:

We know $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$

Does it converge uniformly on $|x| < 1$

Does it converge uniformly on $|x| < 1 - \varepsilon$ for $\varepsilon > 0$

1.)

Pf: Suppose $\sum |a_n||x|^n$ does not converge absolutely $\forall$ X.

Let $R = \sup \{ r \mid \sum |a_n| r^n \text{ converges} \}$

For any $r > R$, diverges by comparison

For any $r < R$, converges by comparison

Example w/ $R = \infty$ $\quad \sum_{n=0}^{\infty} 0 \cdot x^n$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \qquad \forall x \quad \frac{x^{n+1}}{(n+1)!} \Big/ \frac{x^n}{n!} = \frac{x}{n} \to 0 \text{ as } n \to \infty$$

Example w/ $R = 0$ $\quad \sum_{n=0}^{\infty} n!\, x^n$ diverges $\forall$ $|x| > 0$

Example w/ $R = 1$ $\quad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ converges $\forall$ $|x| < 1$, diverges $\forall$ $|x| > 1$

diverges at $x = -R$, $x = R$

does not converge at $x = -1$

diverges at $x = 1$

Example w/ finite R $\quad \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges $\forall$ $|x| \leq 1$, diverges $\forall$ $|x| > 1$

converges at $x = -R$

~~diverges~~ converges at $x = R$

Example w/ finite R $\quad \sum_{n=1}^{\infty} \frac{x^n}{n}$ converges $\forall$ $|x| < 1$, diverges $\forall$ $|x| > 1$

converges at $x = -R$

diverges at $x = R$

converges for $x = -1$

diverges for $x = 1$

Let R be rad. of conv of $\sum_{n=0}^{\infty} a_n x^n$

2) $$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$$

If $\lim |a_n|^{1/n}$ exists then $R = \dfrac{1}{\lim |a_n|^{1/n}}$

~~Pf idea: For $r > \limsup_{n\to\infty} |a_n|^{1/n}$~~

~~For $r < R$ $|a_n| r^n$ s~~

Qualitatively: If coef grow faster than geometrically, $R=0$

If coef grow slower than geometric, $R=\infty$

Radius of conv is given by geometric growth rate

Example: $\sum_{n=0}^{\infty} n!\, x^n$

$a_n = n!$ $\quad \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} (n!)^{1/n} = \lim_{n\to\infty} e^{\frac{1}{n}\log(n!)}$

Stirling's formula $\log n! = n\log n - n + O(\log n)$

$$\lim (n!)^{1/n} = \lim_{n\to\infty} e^{\frac{1}{n}[n\log n - n + O(\log n)]}$$
$$= \lim e^{\log n - 1 + \frac{O(\log n)}{n}}$$
$$= \lim n\, e^{-1 + \frac{O(\log n)}{n}} \to \infty.$$

Activity:

Evaluate $\sum_{n=1}^{\infty} \frac{x^n}{n}$ $\left[= -\log(1-x) \right]$

4)

If derivs of f_n conv unif (and there is a single point that conv) then limit is diffable (and is limit of derivs)

Why must there be a single pt that conv? Can translate up to ∞.

Let $f_n \equiv n$ $\quad f_n' \equiv 0$ $\quad$ so $f_n \not\to f$ though $f_n' \to 0$

Proof gist:
$$f_n(x) = \int_a^x f_n'(\hat{x})\,d\hat{x} + C_n$$

Becau $f_n(x_0)$ converges, get C_n converges C_∞

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int_a^x f_n'(\hat{x})\,d\hat{x} + C_n$$

b/c uniformly (on bdd interval) converging functs allow interchange of limit & integral

$$\lim_{n\to\infty} f_n(x) = \int_a^x \lim_{n\to\infty} f_n'(\hat{x})\,d\hat{x} + C_\infty$$

$$f(x) = \int_a^x g(\hat{x})\,d\hat{x} + C_\infty$$

So $f' = g$.

Activity:

Can interchange

$$\frac{d}{dx}\sum_{n=1}^{\infty}\frac{\sin nx}{n^3} = \sum_{n=1}^{\infty}\frac{\cos nx}{n^2} \quad ?$$

$$\frac{d}{dx}\sum_{n=1}^{\infty}\frac{\sin nx}{n^2} = \sum_{n=1}^{\infty}\frac{\cos nx}{n} \quad ?$$

$$\frac{d}{dx}\sum_{n=1}^{\infty}\frac{\sin nx}{n} = \sum_{n=1}^{\infty}\cos nx \quad ?$$