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Analysis I
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Day 1— Summary — Real Numbers

1. Let $\mathbb{N}$ be the natural numbers, $\mathbb{Z}$ be the integers, $\mathbb{Q}$ be the rationals, and $\mathbb{R}$ be the reals.
2. If $x \in \mathbb{Q}$, then $x = n/m$, for $n, m \in \mathbb{Z}$ and $m \neq 0$.
3. A real number can be defined as an equivalence class of Cauchy sequences of rationals.
4. Some axioms of real numbers:
   (a) $(x + y) + z = x + (y + z)\ \forall x, y, z \in \mathbb{R}$ (additive associativity)
   (b) $0 + x = x + 0\ \forall x \in \mathbb{R}$ (additive identity)
   (c) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $x + y = 0$ (additive inverse)
   (d) $\forall x, y \in R, x + y = y + x$ (additive commutativity)
   (e) $(xy)z = x(yz)\ \forall x, y, z \in \mathbb{R}$ (multiplicative associativity)
   (f) $1x = x\ \forall x \in \mathbb{R}$ (multiplicative identity)
   (g) $\forall x \neq 0, \exists y$ such that $yx = 1$ (multiplicative inverse)
   (h) $xy = yx\ \forall x, y \in \mathbb{R}$ (multiplicative commutativity)
   (i) $x(y + z) = xy + xz\ \forall x, y, z \in \mathbb{R}$ (distributivity)
   (j) Every non-empty set of reals which is bounded from above has a least upper bound. Every non-empty set of reals which is bounded from below has a greatest lower bound. (Completeness axiom)
5. Properties of the reals
   (a) Archimedian property: If $0 \leq x \leq 1/n\ \forall n \in \mathbb{N}$, then $x = 0$
   (b) Density of rationals within the reals: For all $x \in \mathbb{R}$ and $\varepsilon > 0$, there exists $q \in \mathbb{Q}$ such that $|q - x| < \varepsilon$.
   (c) Between two distinct rationals, there is a real. Between two distinct reals, there is a rational.
6. Sequences of real numbers
   (a) The sequence $\{x_n\}_{n=1}^{\infty}$ converges if $\exists\ a \in \mathbb{R}$ such that all $\varepsilon > 0\ \exists N$ such that for all $n \geq N$, $|x_n - a| < \varepsilon$.
   (b) A bounded monotonic sequence converges.