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Quiz 2

Rules: 90 minutes. Open notes, open book, closed electronics.
Please show all of your work. There are 5 problems.

1. (20 points) Let A be the 3×3 symmetric matrix such that $A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is the nearest point to $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ on the plane $x + y = 0$. Is A positive definite? Is A positive semi-definite? Justify your answers.

We begin by finding the eigenvalues & eigenvectors.

Any multiple of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is such that the nearest point on $x+y=0$ is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Hence, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is eigenvector with eigenvalue zero.

Any vector on $x+y=0$ is such that $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Hence there are two linearly independent eigenvectors of eigenvalue 1.

A is not positive ~~semi~~definite because 0 is eigenvalue.
A is positive semidefinite because all eigenvalues are nonnegative.

2. (20 points) Let $A = \tilde{U}\tilde{\Sigma}\tilde{V}^t$ be a reduced SVD of A. The pseudoinverse of A is $A^+ = \tilde{V}(\tilde{\Sigma}^{-1})\tilde{U}^t$.

(a) (10 points) Find the pseudoinverse of

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

(b) (10 points) Use the normal equations to show that the minimizer of $\min_u ||Au - b||^2$ is given by $\hat{u} = A^+ b$.

Hint: You may need the fact that $(BCD)^t = D^t C^t B^t$ for matrices B, C, and D.

a)

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is almost a reduced SVD. The only issue is that $\hat{U}$'s columns are not normal. Divide by $\sqrt{2}$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

b) We want to show that $A^t A \hat{u} = A^t b$

That is, $A^t A A^+ b = A^t b$.

It suffices to show $A^t A A^+ = A^t$.

Computing, $A^t A A^+ = \tilde{V}\tilde{\Sigma}^t \underbrace{\tilde{U}^t \tilde{U}}_{I_{r\times r}} \tilde{\Sigma} \underbrace{\tilde{V}^t \tilde{V}}_{I_{r\times r}} \tilde{\Sigma}^{-1} \tilde{U}^t$

$$= \tilde{V}\tilde{\Sigma}^t \cancel{\tilde{\Sigma}}\cancel{\tilde{\Sigma}^{-1}} \tilde{U}^t$$

$$= \tilde{V}\tilde{\Sigma}^t \tilde{U}^t = A^t.$$

3. (20 points) The fft of the signal x is

$$\hat{x} = \begin{pmatrix} -12 \\ 0 \\ 0 \\ -6i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 6i \\ 0 \\ 0 \end{pmatrix} = 12 \begin{pmatrix} -1 \\ 0 \\ 0 \\ -i/2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ i/2 \\ 0 \\ 0 \end{pmatrix}$$

Find x and sketch its real and imaginary parts.

Suggestion: Instead of writing out all 12 components of x, present a formula for the jth coefficient of x. Simplify your formula by expanding complex exponentials in terms of sines and cosines.

$N = 12$

$x = F_N\left(\frac{1}{N}\hat{x}\right)$

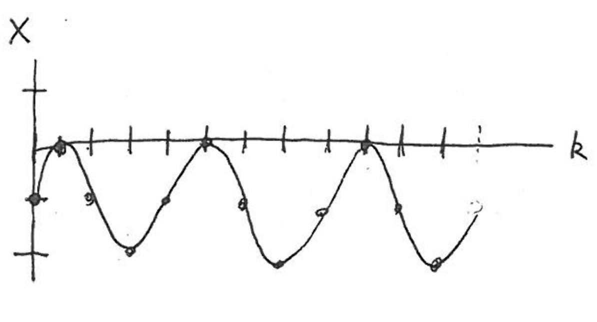
Note $\frac{1}{N}\hat{x}(j) = \begin{cases} -1 & \text{if } j = 0 \\ -i/2 & \text{if } j = 3 \\ i/2 & \text{if } j = -3 \text{ or } N-3 \end{cases}$

$x = -1 \cdot V_0 - \frac{i}{2} \cdot V_3 + \frac{i}{2} V_{-3}$ where V_j is j^{th} Fourier basis vector

Recall $V_j(k) = e^{2\pi i j k/N}$

$$x(k) = -1 - \frac{i}{2} e^{2\pi i \, 3k/12} + \frac{i}{2} e^{2\pi i (-3)k/12}$$

$$= -1 + \frac{e^{2\pi i \, 3k/12} - e^{-2\pi i \, 3k/12}}{2i} = -1 + \sin 2\pi \frac{3k}{12}$$



← 3 oscillations
No imaginary component

4. (20 points)

(a) (10 points) Find the condition number of

$$A = \begin{pmatrix} 1+\varepsilon & 1 \\ 1 & 1+\varepsilon \end{pmatrix}.$$

(b) (10 points) Find a full QR decomposition of

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

a) Rows add up to constant.

$$A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2+\varepsilon\\2+\varepsilon\end{pmatrix}$$

So one eigenvalue is $2+\varepsilon$.

A is symmetric, so other eigenvector is $\perp$ to $\begin{pmatrix}1\\1\end{pmatrix}$

other eigenvector must be $\begin{pmatrix}1\\-1\end{pmatrix}$

$$A\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}\varepsilon\\-\varepsilon\end{pmatrix} = \varepsilon\begin{pmatrix}1\\-1\end{pmatrix}$$

So second eigenvalue is ε.

Condition number is ratio of smallest to largest singular values. In case of, it is ratio of largest to smallest eigenvalues (after absolute value)

$$\text{Cond}(A) = \frac{2+\varepsilon}{\varepsilon}.$$

b) Note columns of B are already orthogonal.
After Gram-Schmidt, they becom $\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}$

We can write down reduced $\hat{Q}\hat{R}$

$$B = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix}\sqrt{2} & 0 \\ 0 & \sqrt{2}\end{pmatrix}$$

To turn this into a full QR, add two other perpendicular vectors to $\hat{Q}$.
and two rows of zeros in $\hat{R}$

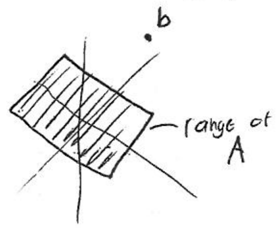
$$B = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix}\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \\ 0 & 0\end{pmatrix}$$

5. (20 points)

   (a) (10 points) Set up (but do not solve) a least squares problem to find the nearest point on the hyperplane $w + x + y + z = 0$ to the point $(1, 2, 3, 4)$.

   (b) (10 points) Repeat, but for the hyperplane $w + x + y + z = 1$.
   Comment: Careful. This hyperplane does not go through the origin.

(a)



Express plane as range of some matrix A.
Need a basis for $\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$ s.t. $w+x+y+z=0$

All such points are of form $\begin{pmatrix} -x-y-z \\ x \\ y \\ z \end{pmatrix}$.

Hence any point on plane is of form

$$x\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

A basis of plane is $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

The least squares problem to find the nearest point to b on range of A is

$$\min_{v} \|Av - b\|^2.$$

If $\hat{v}$ is minimizer, $A\hat{v}$ is nearest point.

Nearest point is $A\hat{v}$ where $\hat{v}$ is minimizer of $\|Av-b\|^2$, where

$$A = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

b) If we shift z down by 1 the problem turns into
Find nearest point on $w+x+y+z=0$ to the point $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$

Observe such a problem is solved as above, but with $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.
Then shift up by 1 in z.

Nearest point is

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + A\hat{v}$$ where $\hat{v}$ is minimizer of $\|Av-b\|^2$

and $$A = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; \& \; b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$$