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Computational Science and Engineering I
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Quiz 1

Rules: Open notes, open book, closed electronics. Time limit: 90 minutes. Please show all of your work.

1. Let z be a nonzero column vector in $\mathbb{R}^n$. Let $A = zz^t$.

   (a) (10 points) What is the rank of A?

   (b) (10 points) Show that the null space of A is the set of all vectors perpendicular to ~~x~~ z.

a) The rank of A is the # independent columns, which is the dimension of the range of A.

Note: the range of A is all multiples of z because $Ax = z\underbrace{z^t x}_{\text{scalar}} = \text{constant} * z$.

So dim range(A) = 1 and hence rank(A) = 1

b) If $Ax = 0$ then $zz^t x = 0$. Note $z^t x$ is the dot product of z with x. It is a scalar. The only multiple of z that is zero corresponds to $z^t x = 0$, hence z perpendicular to x.

2. (20 points) Find an orthonormal basis for the space of points $(x, y, z) \in \mathbb{R}^3$ satisfying

$$\underset{(I)}{x + y + z = 0} \text{ and } \underset{(II)}{- x + y + 2z = 0}.$$

These two planes will intersect along a line, so we seek an orthonormal basis of a line.

By II, $x = y + 2z$. Plugging into 1, we get $y + 2z + y + z = 0$

$\Rightarrow \quad 2y + 3z = 0 \quad \Rightarrow \quad y = -\frac{3}{2} z.$

So if (x, y, z) is on this line, $y = -\frac{3}{2} z$ and $x = y + 2z$.

The general form for a point on the line is

$$\begin{pmatrix} \frac{1}{2} z \\ -\frac{3}{2} z \\ z \end{pmatrix} = z \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{pmatrix}.$$

A basis for the line is given by $\begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{pmatrix}$

To orthonormalize it, we make sure it has length 1.

$$\left\| \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{pmatrix} \right\| = \sqrt{\frac{1}{4} + \frac{9}{4} + \frac{4}{4}} = \frac{\sqrt{14}}{2}$$

An orthonormal basis is $\begin{pmatrix} 1/\sqrt{14} \\ -3/\sqrt{14} \\ 2/\sqrt{14} \end{pmatrix}$.

3. (20 points) Find the rank and null space of B . Justify your answer completely.

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The first 5 columns of B are independent.

If
$$c_1\begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\\0\\0\\0\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\\1\\0\\0\end{pmatrix} + c_4\begin{pmatrix}0\\0\\0\\1\\1\\0\end{pmatrix} + c_5\begin{pmatrix}0\\0\\0\\0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}$$

then first line gives $c_1 = 0$

second line gives $c_1 + c_2 = 0 \Rightarrow c_2 = 0$

third line gives $c_2 + c_3 = 0 \Rightarrow c_3 = 0$

fourth line gives $c_3 + c_4 \Rightarrow c_4 = 0$

fifth line gives $c_4 + c_5 = 0 \Rightarrow c_5 = 0$.

So rank is at least 5.

Note that $B\begin{pmatrix}1\\-1\\1\\-1\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}$ so null space contains $\begin{pmatrix}1\\-1\\1\\-1\\1\\-1\end{pmatrix}$

Hence $\dim \mathrm{Null}(A) \geq 1$.

By rank nullity theorem, rank must be 5 and $\dim \mathrm{Null}(A) = 1$.

$\mathrm{rank}(A) = 5$

$\mathrm{Null}(A) = \mathrm{span}\begin{pmatrix}1\\-1\\1\\-1\\1\\-1\end{pmatrix}$

4. (20 points) Find the LU decomposition of C.

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}$$

$l_{21} = -1$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & +1 \end{pmatrix}$$

$l_{31} = -1$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & \cdot & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & 2 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}$$

$l_{41} \,\&\, l_{51} = -1$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & & 1 & 0 & 0 \\ -1 & & & 1 & 0 \\ -1 & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & 2 \\ 0 & -1 & -1 & 1 & 2 \\ 0 & -1 & -1 & -1 & 2 \end{pmatrix}$$

$l_{32} = l_{42} = l_{52} = -1$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & & 1 & 0 \\ -1 & -1 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -1 & 4 \end{pmatrix}$$

$l_{43} = l_{53} = -1$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & -1 & 8 \end{pmatrix}$$

$l_{54} = -1$

$$\boxed{C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 16 \end{pmatrix}}$$

5. (20 points) A way to find the inverse of a matrix.

(a) (4 points) Suppose B is a 4×4 matrix. For what vector y is $By =$ 2nd column of B + 3rd column of B?

(b) (8 points) Let A be a 4×4 nonsingular matrix. Find the vector b for which the solution to $Ax = b$ is the first column of A^{-1}.

(c) (8 points) Inspired by (b), describe a method for finding the inverse of an $n \times n$ matrix. How many floating point operations are needed to implement this method?
Use the fact that an LU factorization of A involves $\sim \frac{2}{3}n^3$ floating point operations, and that back substitution of a triangular system involves $\sim n^2$ floating point operations.

a) $y = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

b) $Ax = b \Rightarrow x = A^{-1}b$. If $b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then $A^{-1}b$ is the first column of A^{-1}.

c) We will find each column of A^{-1} separately.

The first column given by solving $Ax = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

The second column given by solving $Ax = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

The n^{th} column given by solving $Ax = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$.

This requires n linear solves.

LU factorization of A needs only be done once.

Operation count: $\underbrace{\frac{2}{3}n^3}_{\text{run LU}} + \underbrace{n}_{\text{for each desired column of } A^{-1}} \cdot \big(\underbrace{n^2}_{\text{one back substitution}} + \underbrace{n^2}_{\text{other back substitution}} \big) = \frac{8}{3}n^3$ operations.