25 April 2013
18.085
Computational Science and Engineering I
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Problem Set 6

Due: **2 May 2013** in class.

Print or write out any Matlab input and output.

1. (10 points) Using the Taylor series of $u(x)$ around $x = 0$:

    (a) Show that $\frac{u(\Delta x) - u(0)}{\Delta x}$ is a first-order approximation of $\frac{du}{dx}(0)$.

    (b) Find constants c_0, c_1, c_2 such that $\frac{c_2 u(2\Delta x) + c_1 u(\Delta x) + c_0 u(0)}{\Delta x}$ is a second-order approximation of $\frac{du}{dx}(0)$.

2. (10 points) By hand, find the solution to

$$-\frac{d^2 u}{dx^2} = x$$
$$\frac{du}{dx}(0) = 0 \qquad\qquad (*)$$
$$u(1) = 0$$

3. (20 points) In this problem, you will solve (*) by a finite difference method. Use a grid of N equally spaced points from $x = 0$ to $x = 1$, inclusive. Thus, $\Delta x = \frac{1}{N-1}$.

    (a) Write out by hand the finite difference linear system to be solved in the $N = 5$ case. Approximate $\frac{d^2 u}{dx^2}$ by the standard 3-point discretization. Approximate $\frac{du}{dx}(0)$ by $\frac{u(\Delta x) - u(0)}{\Delta x}$.

    (b) Use Matlab to construct and solve the corresponding linear system for $N = 100$ and $N = 200$.

    (c) What is the maximal error in each of these solutions? Is the method first or second order accurate in Δx.

4. (20 points)

    (a) Show that any solution to (*) satisfies the following weak form

    $$\int_0^1 \frac{du}{dx}\frac{d\phi}{dx} dx = \int_0^1 x\phi(x) dx \text{ for all } \phi \text{ such that } \phi(1) = 0.$$

    Note that $\phi(0)$ is not assumed to be zero.

    (b) Consider a grid $x_i = i\Delta x$ with $\Delta x = \frac{1}{N}$. For $i = 0, \cdots, N-1$, let ϕ_i be the piecewise linear function such that $\phi_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. In the case $N = 5$, draw $\phi_0(x), \cdots, \phi_4(x)$ and write out the linear system arising from a finite element method with these basis functions.

    (c) Write a Matlab program to generate and solve the finite element linear system for the cases $N = 100$ and $N = 200$.

(d) What is the maximal error (over the collection of nodes x_i) in each of these two solutions?

5. (20 Points) Consider the 2π periodic boundary value problem

$$-\frac{d^2u}{dx^2}(x) = f(x)$$
$$u(-\pi) = u(\pi)$$
$$\frac{du}{dx}(-\pi) = \frac{du}{dx}(\pi)$$

where $f(x) = \delta(x - \pi/2) - \delta(x + \pi/2)$.

(a) Sketch a solution to the boundary value problem.

(b) Expand f as a Fourier series: $f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$. Find $\hat{f}(n)$.

(c) Expand u as a Fourier series: $u(x) = \sum_{n=-\infty}^{\infty} \hat{u}(n)e^{inx}$. Compute $\frac{d^2u}{dx^2}$ and use the differential equation to solve for $\hat{u}(n)$. Argue that $\hat{u}(0)$ can be anything.

(d) Set $\hat{u}(0) = 0$. Use Matlab to plot $\sum_{n=-20}^{20} \hat{u}(n)e^{inx}$ for $-\pi \le x \le \pi$.