Computational Science and Engineering
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Problem Set 4 Solutions

1)

(a) Use MATLAB:

```
x=linspace(-1,1,50)';
A=vander(x);
y=1/3*x;
Ccomputed=A\y;
Cactual=zeros(50,1);
Cactual(49)=1/3;

%calculate the fractional error
FracError=norm(Ccomputed-Cactual)/norm(Cactual);
```

The fractional error is 3.1656×10^5 (quite a large fraction!).

(b) Use MATLAB:

```
%calculate the condition number of A
K=cond(A);
```

The condition number of A is 3.9137×10^{18}.

We can compute the expected fractional error of the coefficients by multiplying the condition number by the expected errors in the variables as stored by MATLAB.

Determine the rounding error:

```
a=1/10
sprintf('%0.33f',a)
ans = 0.100000000000000010000000000000000
```

The rounding error is on the order of 10^{-16} which means that the expected fractional error would be 10^2. Since this is smaller than 10^5, this implies that there are other sources of error leading to the large fractional error.

2) (a) The rows add up to 1 which means that

$$A\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}.$$

Hence $\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector of eigenvalue 1.

Because A is symmetric, it has an orthonormal basis of eigenvectors. Thus $\begin{bmatrix}1\\-1\end{bmatrix}$must be an eigenvector.

$$A\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1/2\\-1/2\end{bmatrix}.$$

Hence $\begin{bmatrix}1\\-1\end{bmatrix}$ is an eigenvector of eigenvalue ½.

Normalizing the eigenvectors and writing A in the form of eigenvalue decomposition we get:

$$A=\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1/2\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$$

(b)

$A = Q\Lambda Q^T$

$A^n = Q\Lambda Q^T Q\Lambda Q^T \dots Q\Lambda Q^T \ (n\ times)$

Because Q is orthonormal, $Q^T Q = I$ so $A^n = Q\Lambda^n Q^T$ where

$$\Lambda^n = \begin{bmatrix} 1 & 0 \\ 0 & (1/2)^n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2^n \end{bmatrix}.$$

As $n \to \infty, A^n \to \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$

3)

(a) Find the full singular value decomposition of

$$A = \begin{bmatrix} 0 & 4 \\ 0 & 0 \\ 2 & 0 \end{bmatrix}.$$

The reduced SVD of A is

$$A = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} - & v_1 & - \\ - & v_2 & - \end{bmatrix}$$

Where u_1 and u_2 are 3x1 and v_1 and v_2 are 1x2.

By rearranging $A = U\Sigma V^T$ we get $AV = U\Sigma$, so the columns of $U_{reduced}$ must be in the range of A.

The range of A spans $\begin{bmatrix} 0 & 4 \\ 0 & 0 \\ 2 & 0 \end{bmatrix}$ which is normalized to

$$U_{reduced} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Similarly, we can say that $A^T U = \Sigma V$ so the columns of$V_{reduced}$must be in the range of A^T. The range of A^T spans $\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ which is normalized to

$$V_{reduced} = V_{reduced}{}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We can find Σ using $AV = U\Sigma$:

$$A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

So

$$\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Our reduced SVD is thus

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

To make this a full SVD we add the remaining orthonormal column to U and rearrange the columns to ascend from largest to smallest singular value to get

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) The range of B is given by the columns of U:

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

The null space of $B$is given by the rows of V^T that correspond to the singular values of zero:

$$\begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}.$$

(c)

Since C is a reflection matrix, vectors in the plane of reflection are eigenvectors of eigenvalue 1. For example, (1, -1, 0) and (1, 0, -1) can be chosen as eigenvectors.

Since the reflection matrix is symmetrical, the third eigenvector is perpendicular to the plane and of eigenvalue -1. (1, 1, 1) can be chosen as the third eigenvector.

These three eigenvectors can be put in a matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

and made orthonormal using Gram-Schmidt

$$S = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -\sqrt{2/3} \end{bmatrix}.$$

The full eigenvalue decomposition and SVD:

$$C = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -\sqrt{2/3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -\sqrt{2/3} \end{bmatrix}$$

4)

$$v_j(k) = e^{2\pi ijk/N}$$

To show $v_j$is an eigenvector, show $Av_j(k) = \lambda_j v_j(k)$:

$$Av_j(k) = \sum_m A_{km} v_j(k)$$

$$= -v_j(k-1) + 2v_j(k) - v_j(k+1)$$

$$= -e^{2\pi ij(k-1)/N} + 2e^{2\pi ijk/N} - e^{2\pi ij(k+1)/N}$$

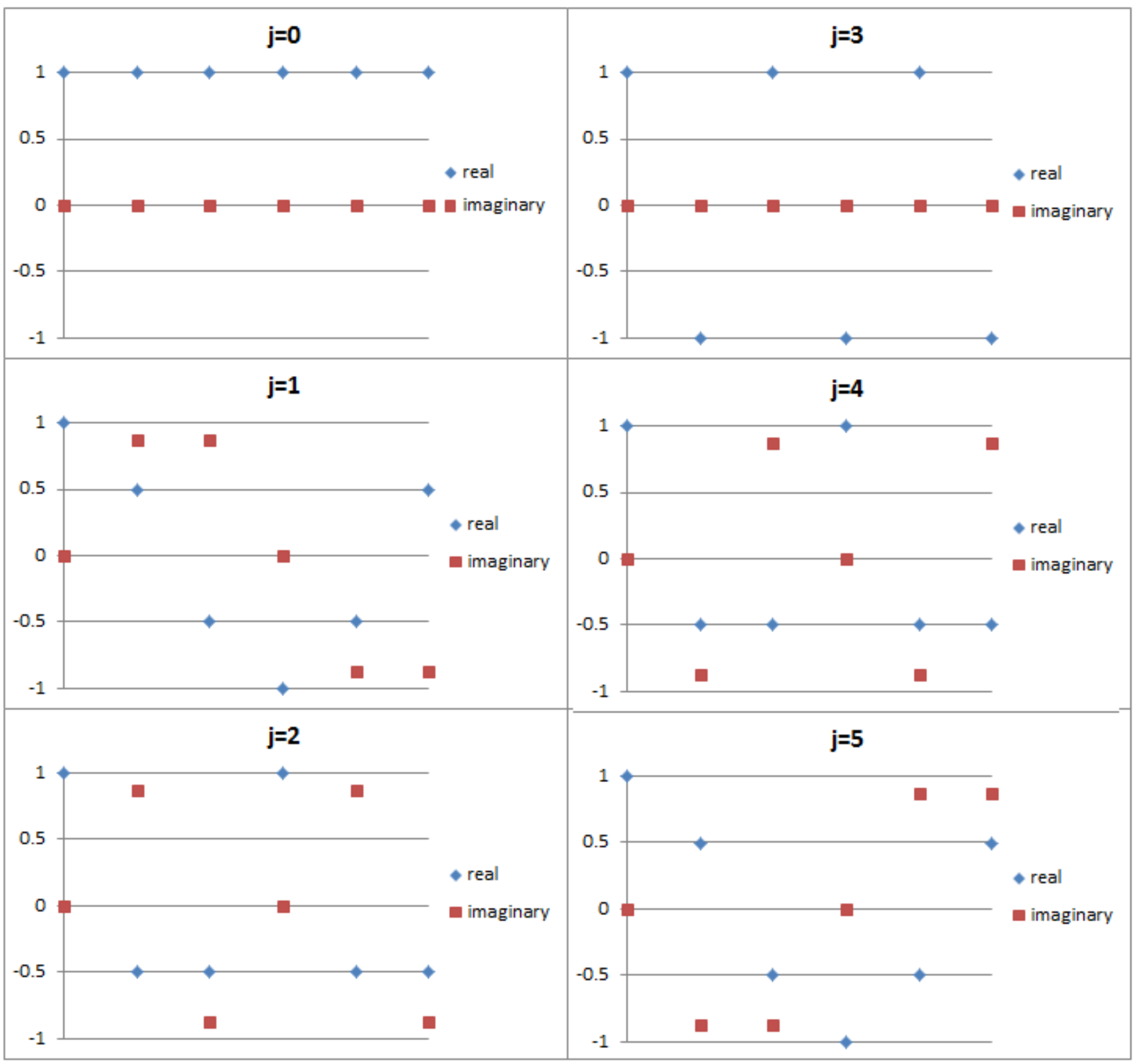
$$= e^{2\pi ijk/N}\left(-e^{-2\pi ij/N} + 2 + e^{2\pi ij/N}\right)$$

$$= e^{2\pi ijk/N}\left(2 - 2cos(2\pi j/N)\right).$$

Hence v_j is an eigenvector of eigenvalue $2 - 2cos(2\pi j/N)$.

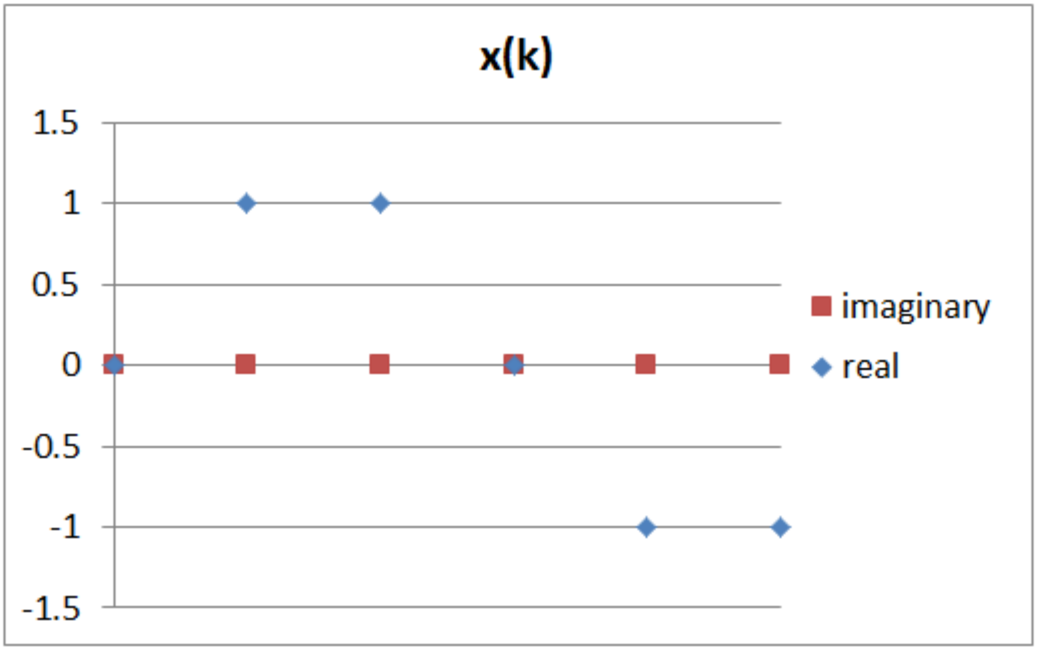
5)
(a)

$$F_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2}+\frac{\sqrt{3}}{2}i & -\frac{1}{2}+\frac{\sqrt{3}}{2}i & -1 & -\frac{1}{2}-\frac{\sqrt{3}}{2}i & \frac{1}{2}-\frac{\sqrt{3}}{2}i \\ 1 & -\frac{1}{2}+\frac{\sqrt{3}}{2}i & -\frac{1}{2}-\frac{\sqrt{3}}{2}i & 1 & -\frac{1}{2}+\frac{\sqrt{3}}{2}i & -\frac{1}{2}-\frac{\sqrt{3}}{2}i \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{2}-\frac{\sqrt{3}}{2}i & -\frac{1}{2}+\frac{\sqrt{3}}{2}i & 1 & -\frac{1}{2}-\frac{\sqrt{3}}{2}i & -\frac{1}{2}+\frac{\sqrt{3}}{2}i \\ 1 & \frac{1}{2}-\frac{\sqrt{3}}{2}i & -\frac{1}{2}-\frac{\sqrt{3}}{2}i & -1 & -\frac{1}{2}+\frac{\sqrt{3}}{2}i & \frac{1}{2}+\frac{\sqrt{3}}{2}i \end{bmatrix}$$



(b) (i) $x = [0 \quad 1 \quad 1 \quad 0 \quad -1 \quad -1]^T$

$x(k)$ is graphed below with respect to k (row number). We see that $x(k)$ could be described as a combination of the columns of F_6 as illustrated in the plots above.



We can calculate $x(k)$ from the shape of its plot:

$$x(k) = \frac{\sin\left(\frac{2\pi k}{6}\right)}{\sqrt{3}/2}$$

$$= \frac{2}{\sqrt{3}} sin\left(\frac{2\pi k}{6}\right)$$

$$= \frac{2}{\sqrt{3}}\left(\frac{e^{2\pi i k/6} - e^{-2\pi i k/6}}{2i}\right)$$

$$= 1/\sqrt{3}i\left(e^{2\pi i(1)k/6}\right) - 1/\sqrt{3}i\left(e^{2\pi i k(5)/6}\right)$$

From $F_6 \frac{1}{6}\hat{x} = x$:

$$\frac{1}{6}\hat{x} = \begin{bmatrix} 0 \\ 1/\sqrt{3}i \\ 0 \\ 0 \\ 0 \\ -1/\sqrt{3}i \end{bmatrix} \rightarrow \hat{x} = \begin{bmatrix} 0 \\ -2\sqrt{3}i \\ 0 \\ 0 \\ 0 \\ 2\sqrt{3}i \end{bmatrix}.$$

Confirm with MATLAB:
```
fft([0 1 1 0 -1 -1]')
ans =
         0
         0 - 3.4641i
         0
         0
         0
         0 + 3.4641i
```

(ii) $x = [1 \quad 2 \quad 2 \quad 1 \quad 0 \quad 0]^T$

This x is the same as the one from part (i) except that a vector of ones has been added to it. This is reflected in $\hat{x}$ by putting a one in the first position (which is then multiplied by N=6):

$$\hat{x} = \begin{bmatrix} 6 \\ -2\sqrt{3}i \\ 0 \\ 0 \\ 0 \\ 2\sqrt{3}i \end{bmatrix}$$

Confirm with MATLAB:
```
fft([1 2 2 1 0 0]')
ans =
   6.0000
         0 - 3.4641i
         0
         0
         0
         0 + 3.4641i
```

(iii) $x = [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]^T$

Calculated directly:

$$\hat{x} = \bar{F}_6 x = \bar{F}_6 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \left[e^{-2\pi i \frac{k}{6}}\right]_{\{k=0,\ldots 5\}} = \begin{bmatrix} 1 \\ \frac{1}{2} - \frac{\sqrt{3}}{2} i \\ -\frac{1}{2} - \frac{\sqrt{3}}{2} i \\ -1 \\ -\frac{1}{2} + \frac{\sqrt{3}}{2} i \\ \frac{1}{2} + \frac{\sqrt{3}}{2} i \end{bmatrix}$$

which is the first (zero indexed) column of $\bar{F}_6$ and equals the complex conjugate of the Fourier basis element v_1.

Confirm with MATLAB:
```
fft([0 1 0 0 0 0]')
ans =
   1.0000
   0.5000 - 0.8660i
  -0.5000 - 0.8660i
```

```
  -1.0000
  -0.5000 + 0.8660i
   0.5000 + 0.8660i
```

6) Using MATLAB:

```
note=wavread('single_note_piano');
[Y,I]=max(fft(note));
```

Result:

```
Y =  3.7114e+003 +2.6170e+003i
I =  294 %position (1-indexed)
```

This means that the wave oscillates 293 times in 66150 index positions. The sampling rate is 44100 samples/second so the wave oscillates at 293 times in 1.5seconds of 195.3 Hz.

A piano table gives this as G3.