Computational Science and Engineering
Amelia Servi
March 26, 2013

Problem Set 3 Solutions [Revised]

1) Find the QR factorization of

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix}.$$

Using Gram-Schmidt:

$$q_1 = \frac{a_1}{|a_1|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \div 1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$B = a_2 - (q_1^T a_2) q_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{B}{|B|} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \div \sqrt{2} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$C = a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0 \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$q_3 = \frac{C}{|C|} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \div \sqrt{2} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

Thus

$$Q = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

$$R = Q^T A = \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

Together:

$$A = QR = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}.$$

Now solve

$$Ax = QRx = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}.$$

$$Rx = Q^T \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix}.$$

Using substitution:

$$Rx = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix}.$$

$$x_3 = 2$$

$$x_2 = 2$$

$$x_1 + x_2 + x_3 = 5 \rightarrow x_1 = 1$$

Result:

$$x = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

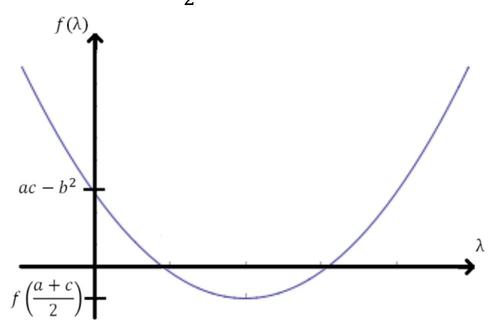
2) (a) Show that if $a > 0$ and $ac - b^2 > 0$, then $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive-definite.

We show both eigenvalues positive if $a > 0$ and $ac - b^2 > 0$. Observe $c > 0$ as otherwise$ac - b^2 < 0$.

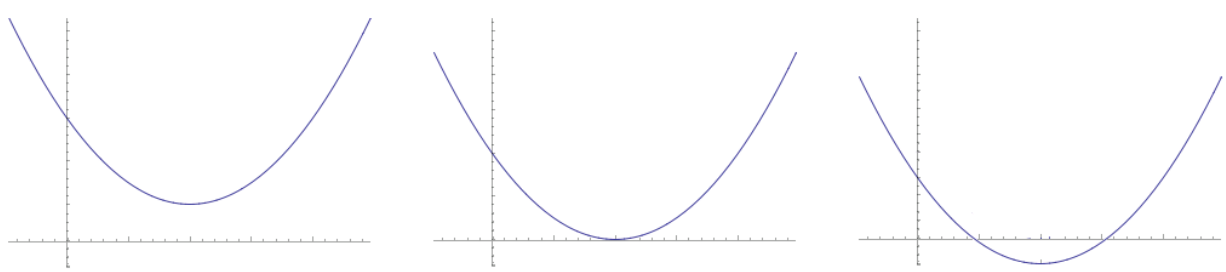
Eigenvalues are given by roots at

$$(\lambda - a)(\lambda - c) - b^2 = \lambda^2 - (a + c)\lambda + ac - b^2 = 0.$$

This is a parabola with minimum value at $\lambda = \frac{a+c}{2}$ and a positive value $ac - b^2$ at $\lambda = 0$.



Three cases:



min value > 0   min value = 0   min value <0

Min value > 0: Roots are complex, matrix is not positive-definite.

Min value = 0, min value <0: Roots are positive and the matrix is positive-definite.

Suffices to show that the minimum value is less than or equal to zero:

Plug in $\lambda = \frac{a+c}{2}$ into $f(\lambda) = \lambda^2 - (a+c)\lambda + ac - b^2$.

$$f\left(\frac{a+c}{2}\right) = \left(\frac{a+c}{2}\right)^2 - \frac{(a+c)^2}{2} + ac - b^2 = -\frac{(a+c)^2}{4} + ac - b^2 = -\frac{(a-c)^2}{4} - b^2 \leq 0.$$

(b)

If $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ is positive-definite, then

$$x^T A x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq 0.$$

If $x_3 = 0$, then $x^T A x$ reduces to $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0$.

Therefore $B = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$is also positive-definite.

(c) Find an x such that $x^T A x < 0$.

For example: $x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Test: $\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = -1$

A is not positive semi-definite.

3)

(a)

$$A^T A \hat{u} = A^T b$$
$$A^T A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, A^T b = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Normal equations:

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \hat{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$
$$\hat{u} = \begin{bmatrix} 4/3 \\ 3/2 \end{bmatrix}.$$

(b) Gram Schmidt of $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ gives $\tilde{Q} = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$, $\tilde{R} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$.

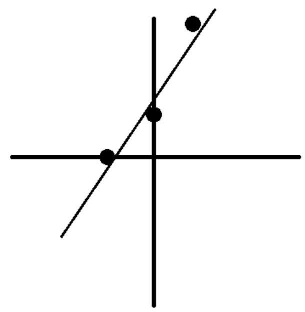
$$\tilde{R}\hat{u} = \tilde{Q}^T b$$
$$\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \hat{u} = \begin{bmatrix} 4/\sqrt{3} \\ 3/\sqrt{2} \end{bmatrix}$$
$$\hat{u} = \begin{bmatrix} 4/3 \\ 3/2 \end{bmatrix}.$$

(c)

Condition number of $A^T A$ is $\frac{\lambda_{max}}{\lambda_{min}} = \frac{3}{2}$ (worse).

Condition number of $\tilde{R}$ is $\frac{\lambda_{max}}{\lambda_{min}} = \sqrt{\frac{3}{2}}$.

(d) Find the best fit line through (-1, 0), (0, 1), (1, 3):

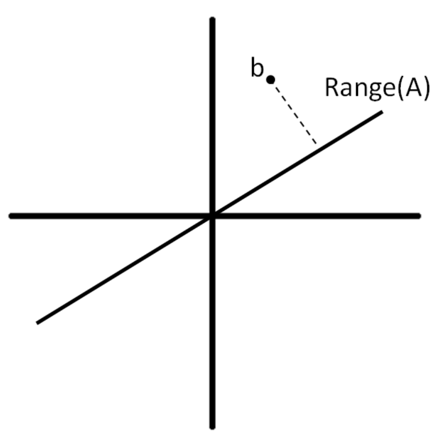


4)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\hat{u} = A \backslash b = \begin{bmatrix} 3.5 \\ -1.5 \\ 0 \end{bmatrix}$$

$A\hat{u} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 5 \\ 5 \\ 5 \end{bmatrix}$ is the nearest point to b.



5)

(a) Yes, $(xx^T)^T = (x^T)^T x^T = xx^T$.

(b) The range of A is the line spanned by x. Observe x is an eigenvector of eigenvalue $|x|^2$:

$$Ax = xx^T x = x(x^T x) = x|x|^2 = 55x$$

The rank of A is 1. Dimension of the null space is 4. Any vector in the null space is an eigenvector of eigenvalue 0. Hence we need only find a basis of the null space.

The null space is the set of all vectors perpendicular to the rows of A. All rows are multiples of x^T.

Here we find all vectors perpendicular to $\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$

$$\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} \cdot \begin{bmatrix}v\\w\\x\\y\\z\end{bmatrix} = 0 \ \rightarrow v + 2w + 3x + 4y + 5z = 0.$$

5 variables, 1 constraint.

All solutions are of the form $\begin{bmatrix}-2w-3x-4y-5z\\w\\x\\y\\z\end{bmatrix} = w\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} + x\begin{bmatrix}-3\\0\\1\\0\\0\end{bmatrix} + y\begin{bmatrix}-4\\0\\0\\1\\0\end{bmatrix} + z\begin{bmatrix}-5\\0\\0\\0\\1\end{bmatrix}$.

An eigenbasis is:

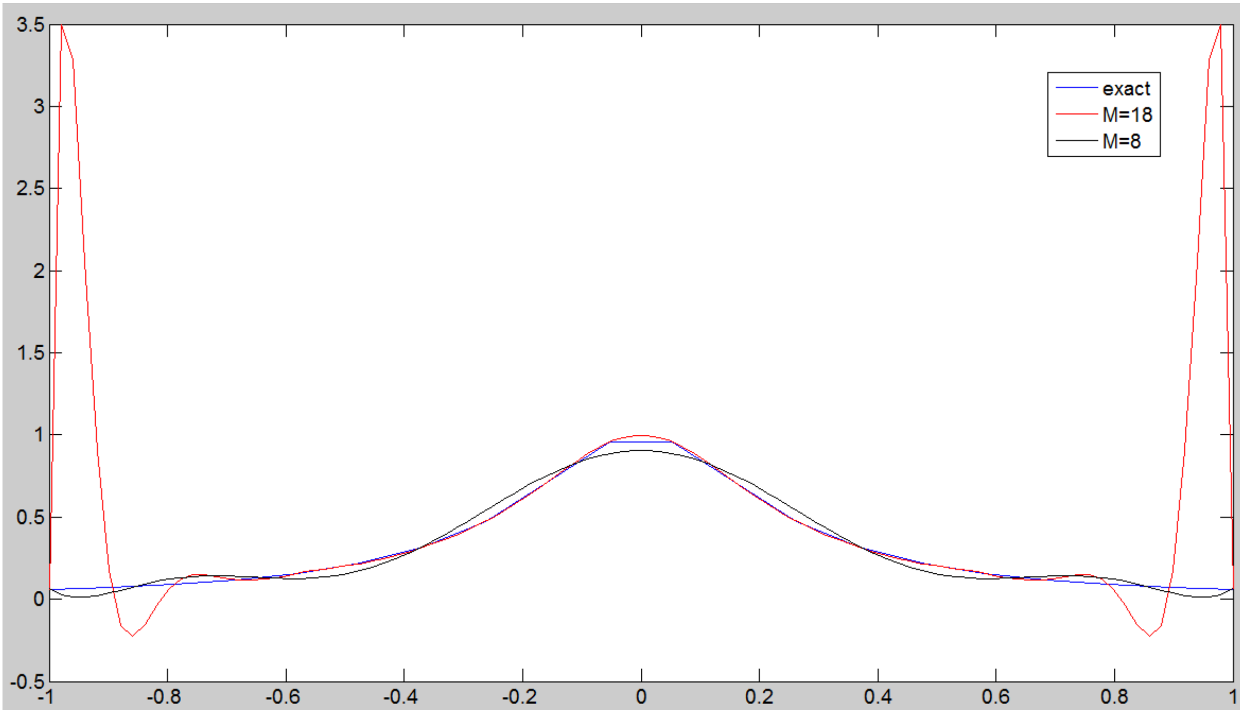
$$\lambda = 55: \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}, \lambda = 0: \begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}\begin{bmatrix}-3\\0\\1\\0\\0\end{bmatrix}\begin{bmatrix}-4\\0\\0\\1\\0\end{bmatrix}\begin{bmatrix}-5\\0\\0\\0\\1\end{bmatrix}.$$

6) (a)

```
% This script finds the best fit Mth degree polynomial to
% y(x) = 1/(1+16x^2) sampled at N equispaced points from -1 to 1, inclusive
% It plots the result on a grid 5 times finer than the data samples

N = 20;    % Number of grid points
M = 18;     % Degree of polyomial
x = linspace(-1, 1, N)';
y = 1 ./ (1 + 16 * x.^2);

A = vander(x);
A = A(:, end-M:end);
u = A \ y;

x_dense = linspace(-1, 1, 5*N);
A_dense = vander(x_dense);
A_dense = A_dense(:, end-M:end);

figure;plot(x, y);
hold on;plot(x_dense, A_dense * u, 'r');
```

(b) See above code using N=20, M=8.

(c)



(d)

The 18th degree polynomial has less residual because all 8th degree polynomials are also 18th degree polynomials with zeros for the large powers.

The 18th degree polynomial is great at approximating the data in the middle of the domain.

The 8th degree polynomial is better at approximating the data near the boundary.