11 March 2013
18.085
Computational Science and Engineering I
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Problem Set 3

Due: **21 March 2013** in class.
Unless otherwise stated, feel free to use Matlab's "\" to perform least squares computations.
Print or write out the Matlab input and output.

1. (10 points) By hand, find the QR factorization of

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

and use it to solve $Ax = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}$

2. (20 points)

   (a) Show that $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive-definite if $a > 0$ and $ac - b^2 > 0$.

   (b) If $A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$ is positive semi-definite, show that $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ is positive semi-definite.

   (c) Is $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ positive semi-definite? Justify your answer.

3. (20 points) Consider the least squares problem: $\min_u \|Au - b\|^2$, where

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

   (a) Form the normal equations and solve for the minimizer $\hat{u}$ by hand.

   (b) Find the reduced QR factorization, $A = \tilde{Q}\tilde{R}$, by hand. Use it to solve for the minimizer $\hat{u}$.

   (c) Read off the condition numbers of A^tA (from part a) and $\tilde{R}$ (from part b)? Which one is worse?

   (d) What best-fit problem does this least squares problem correspond to? Sketch the data points and the best-fit curve.

4. (10 points) Find the nearest point to $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}^t$ on the hyperplane spanned by

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

5. (10 points) Let $x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$.

(a) Is $A = xx^t$ symmetric?

(b) Find a basis of eigenvectors of $A = xx^t$. What are the corresponding eigenvalues? You do not need to make the basis orthonormal.

Hint: Think about the range and null space of xx^t. Think about how null spaces are related to eigenvectors/eigenvalues. Do not try to use the determinant formula for finding eigenvalues.

6. (20 points) Let $x_1, \cdots, x_{20}$ be 20 equally spaced points from -1 to 1, inclusive. Let $y_i = \frac{1}{1+16x_i^2}$.

(a) Find the best-fit 18th degree polynomial to the data points (x_i, y_i).

(b) Find the best-fit 8th degree polynomial to the same data points.

(c) Plot the data points and these two polynomials for $-1 \leq x \leq 1$. For the plot, use a fine enough grid spacing that you can tell what is really going on.

(d) Which polynomial has a smaller square residual with the data? Comment on which approximates the data better. Your answer should be somewhat nuanced.