14 February 2013
Computational Science and Engineering I
Paul E. Hand
hand@math.mit.edu

Problem Set 2

Due: **28 February 2013** in class

1. (10 points) Show that

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & 0 & 1 \end{pmatrix} \text{ is the inverse of } S = \begin{pmatrix} 1 & 0 & 0 \\ -\ell_{21} & 1 & 0 \\ -\ell_{31} & 0 & 1 \end{pmatrix}$$

2. (10 points) By trial and error, find examples of 2 by 2 matrices such that

   (a) $AB \neq BA$

   (b) $A^2 = -I$, with only real entries in A

   (c) $B^2 = 0$, with no zeros in B

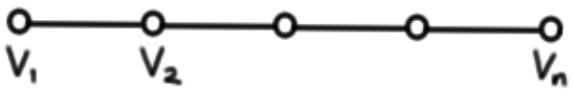
3. (10 points) By hand, factor the matrix $A = LU$, where L is lower triangular, U is upper triangular, and

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

4. (10 points) Use back substitution twice by hand to solve $LUx = f$, where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, U = \begin{pmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 7 \end{pmatrix}, \text{ and } f = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$$

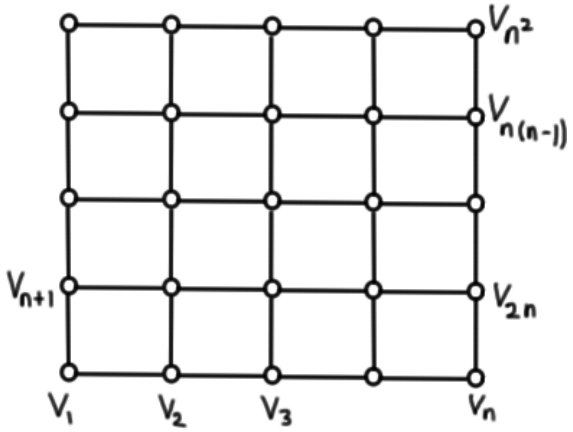
5. (20 points) Consider a line of n nodes, each connected to its neighbors by a resistor of resistance R. At the first node, potential is set to 1. At the nth node, potential is set to 0.



   (a) Write down n equations relating $v_1, v_2, \cdots, v_n$. For $n = 5$, write out by hand the equations in the form $Ax = b$.

   (b) Write a Matlab program that, for arbitrary n, forms A and b and solves for x. Solve $Ax = b$ in the case of n =10,000. What is the computed value of v_{5000}? Provide 6 digits. How long does it take to solve $Ax = b$ in this case? Ignore the time it takes to build the matrix A. Print out your Matlab code.

6. (20 points) Consider the 2d lattice of points from $(1, 1)$ to (n, n). Each is connected to its neighbors by a resistor of resistance R. At the first node $v_1 = 1$. At the last node, v_{n^2} = 0.



(a) In the $n = 3$ case, write out by hand the 9 linear equations in the form $Ax = b$.

(b) Write a Matlab program that, for arbitrary n, forms A and b and solves for x. Solve $Ax = b$ in the case of $n = 100$. What is the computed value of v_{50}? Provide 6 digits. How long does it take to solve $Ax = b$ in this case? Ignore the time it takes to build the matrix A. Print out your Matlab code.

(c) Based on the results of 5b and 6b: is a one-dimensional problem involving 10,000 nodes more, less, or equally expensive as a two-dimensional problem involving 10,000 nodes?