18.085

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Computational Science and Engineering I
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Quiz 1

Rules: Open notes, open book, closed electronics. Please show all of your work.

1. (20 points) Find all the points in the null space of A.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Use back substitution

$e = 0$ (5th row)
$e = 0$ (4th row)
$d = 0$ (3rd row)
$b + c = 0$ (2nd row) $c = -b$
$a + b = 0$ (1st row) $a = -b$.

Any value of b is permissable.

All points are given by $\begin{pmatrix} -b \\ b \\ -b \\ 0 \\ 0 \end{pmatrix} = b \begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

2. (20 points) Find a basis for the space of points (w, x, y, z) satisfying

$$w + 2x + 3y + 4z = 0$$

4 unknowns. 1 eqn.

View eqn as constraint on w given x, y, z.

The set of all points satisfying $w+2x+3y+4z=0$

is

$$\begin{pmatrix} -2x-3y-4z \\ x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

A ~~the~~ basis is $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

We observe these vectors are independent:

$$c_1\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $c_1 = 0$ (second row)

$c_2 = 0$ (third row)

$c_3 = 0$ (4^{th} row)

3. (20 points) Find the rank of B. Justify your answer completely.

$$B = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \\ \begin{matrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{matrix} \end{array}$$

Observe $b_5 = b_1 + b_4 - b_2$

$b_6 = b_1 + b_4 - b_3$.

The first four columns are linearly independent

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow c_4 = 0$ (6^{th} row)

$c_3 + c_4 = 0$ (5^{th} row) $\Rightarrow c_3 = 0$

$c_2 + c_3 + c_4 = 0$ (4^{th} row) $\Rightarrow c_2 = 0$

$c_1 + c_2 + c_3 = 0$ (3^{rd} row) $\Rightarrow c_1 = 0$.

Only linear combination of b_1, b_2, b_3, b_4 that is 0 is the trivial combination.

Rank is 4.

4. (20 points) Find the LU decomposition of C.

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}$$

$l_{21} = -1$
$l_{31} = -1$
$l_{41} = -1$
$l_{51} = -1$

Subtract -1 * first row from all others

$$C = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ -1 & & 1 & & \\ -1 & & & 1 & \\ -1 & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & 2 \\ 0 & -1 & -1 & 1 & 2 \\ 0 & -1 & -1 & -1 & 2 \end{pmatrix}$$

$l_{32} = -1$
$l_{42} = -1$
$l_{52} = -1$

Subtract (-1) * 2^{nd} row from 3^{rd}–5^{th}

$$C = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ -1 & -1 & 1 & & \\ -1 & -1 & & 1 & \\ -1 & -1 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -1 & 4 \end{pmatrix}$$

$l_{43} = -1$
$l_{53} = -1$

Subtract (-1) * 3^{rd} row from 4^{th}–5^{th}

$$C = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ -1 & -1 & 1 & & \\ -1 & -1 & -1 & 1 & \\ -1 & -1 & -1 & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & -1 & 8 \end{pmatrix}$$

$l_{54} = -1$

Subtract (-1) * 4^{th} row from 5^{th}

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 16 \end{pmatrix}$$

5. (20 points) A way to find the inverse of a matrix.

(a) (4 points) Suppose B is a 4×4 matrix. For what vector y is $By =$ 2nd column of B + 3rd column of B?

(b) (8 points) Let A be a 4×4 nonsingular matrix. The solution to the linear system $Ax = b$ is given by $x = A^{-1}b$. If you want to find the 4th column of A^{-1}, what linear system could you solve?

(c) (8 points) Inspired by (b), describe a method for finding the inverse of an $n \times n$ matrix. How many $n \times n$ linear systems would you need to solve?

a) $y = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

b) $x = A^{-1}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is 4th column of A^{-1}.

This is, to find the 4th column of A^{-1}, we could solve $Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

c) Find each column of A^{-1} individually.

The k^{th} column is given by solving $Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ ← 1 in k^{th} position

This method requires n linear solves of $Ax = e_k$.

[Comment: when solving by LU, only need to factorize $A = LU$ once!]