The Ohio State University

Development of a Neonatal Tactile Connectome using Graph Theory for the Prediction of Neurobehavioural Adaptation in Early Childhood

Submitted in Partial Fulfillment of the Requirements for the Degree
Bachelor of Technology in Computer Science and Engineering

By

Girik Malik

Undergraduate Program in Computer Science and Engineering

Shiv Nadar University
India

2017

Advisors

Nathalie L. Maitre, MD, PhD

Andrzej Kloczkowski, PhD, DSc

Irina A. Buhimschi, MD

Approval

This report titled *Development of a Neonatal Tactile Connectome using Graph Theory for the prediction of Neurobehavioural adaptation in early childhood* by *Girik Malik* is approved for the degree of B. Tech in Computer Science and Engineering.

Project Adviser(s)

(Nathalie L. Maitre) Principal Adviser
Date

(Andrzej Kloczkowski)
Date

(Irina A. Buhimschi)
Date

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas or words have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that any violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

(Girik Malik)
Date

Abstract

We demonstrate for the first time differences in functional and effective tactile connectivity in full-term and preterm infants using complex network analysis. Our baseline network studied the resting state connections of neural signal amplitude as measured using a high-density electroencephalography (EEG) array, prior to stimulation of the infant hand with a tactile stimulus. We compared event-related potential (ERP) data from EEG measurements between 54 full-term babies (born after a typical 40 weeks pregnancy period) and 32 preterm infants (born prior to 31 weeks gestation). Measurements were performed both in the resting state and in the functional state in response to the tactile stimulus (air-puffs delivered by a custom-made apparatus) using a high-density array of 128 electrodes embedded in a soft net.

Using network analysis of the ERP data, we were able to identify an organization into smaller communities, using the louvain method of community detection, based on modularity maximization. Because of the potential criticisms of modularity maximization being bound by the problem of resolution limit, we further extracted the connected components from the derived network and found signatures of both, resting state and the functional state connectomes in pre-specified windows after a tactile stimulus.

Further we referenced individual infants to the full-term grand averages of the respective time windows to specifiy common and uncommon nodes . The results of the analysis give clear markers of changes in functional and resting networks for both term and preterm infants. We also showed that infant brain networks are not scale-free and that the degree distributions instead follow exponentially truncated powerlaw.

Dedicated to my Grand Father

Acknowledgments

This thesis would have been impossible without the support and mentorship of my advisors, Prof. Nathalie L. Maitre, Prof. Andrzej Kloczkowski and Prof. Irina A. Buhimschi. They have continued to invite me to The Research Intitute at Nationwide Children's Hospital since 2015 and have always extended a very generous support. They do remain a source of inspiration for me as I am constantly surprised by their amazing intelligence, infinite energy, boundless optimism and genuine friendliness. I appreciate the support I have received from each of them and how much I have learned from them as an individual, not only about great science they do but also about *life* in general.

I would like to acknowledge the support of my roots from Shiv Nadar University, some amazing people that I have met since 2013 and have had fruitful discussions with, which have helped me understand things and look at them from different perspectives. I sincerely thank my professors for being supportive of my work and providing adequate guidance and resources for enabling me to do things which I felt were of more interest and relevance to me. In the past few years, I have received immense support, not only from the computer science department, which made this visit to Columbus, OH possible, but also from the School of Natural Sciences, specially the Department of Mathematics and Big Data and Analytics Center. I would also like to thank my friends at SNU, Kumar Ashutosh, Himanshu Sharma, Sabyasachi Mukherjee, Pankaj Kumar (now at Skolkovo Institute of Science and Technology) and others, for our discussions and their unbiased opinions and suggestions

about various problems and the new ones that I have got to know about for which I have had to scratch my mind.

A considerable mention is also deserved by my friends and colleagues Aisha Altaf Trumboo and Mumin Hakim, here at Nationwide Children's Hospital, who have always supported and helped execute my decisions.

I would always remain indebted to my parents and my two sisters for their constant support and for being my source of motivation.

This acknowledgment would remain incomplete without thanking the invisible supernatural force, the *Almighty*, who gave me the strength and courage to do what I wanted to, guided my decisions in the right direction and made somethings impossible so that I can look at other things that would do more good to me.

This work was supported by Nationwide Children's Hospital's grants 20102216, 20067016 and Pedal-with-Pete's grant 82131216.

Contents

List of Tables

Table **Page**

List of Figures

Figure **Page**

Chapter 1: Introduction

About 15 million babies are born preterm each year and over a million of these babies do not survive [12] [38]. Neurodevelopmental problems are significantly more common in children born preterm. Studies show that preterms are at a higher risk of developing autism, attention deficit / hyperactivity disorders and reduced cognition at school age [11], have emotional disorders [35], are at a higher risk of manifesting neurological abnormalities [41] and of educational underachievement [15] . Neurodevelopmental disabilities in infants have been investigated using different approaches including Graph theoretic perspective, in particular, connectivity and complex networks using data from non-invasive techniques. These novel perspectives have given new insights into disorders and how they alter the neural networks.

Small World networks have been detected in the adult human brain [51] as well as in children aged from 7 years and older [23] [49]. Previous studies have used fMRI to detect the presence and alterations of small world networks within the functional brain networks in patients with schizophrenia [37]. The topological structure of the functional network of the infant brain may also be a small-world network [26] . An increase in average clustering and path length and decrease in weight dispersion within the network may indicate brain maturation characterized by a shift from random to more organized small-world functional networks [14].

Resting state functional connectivity derived from fMRI data demonstrates the predominance of anatomical proximity in infants and functional connectivity in adults, when examined using graph connectivity, community detection, and clustering coefficients. These studies also suggest the presence of a sophisticated information processing system established in early stages of brain development, though they are limited by the resolution of the fMRI methodology. Functional MRI has been successfully used in the studies of early biomarkers of childhood-onset schizophrenia [3]. Brain networks in two-year-olds have demonstrated higher modularity than in younger babies or adults [25]. The identification of hubs and the importance of network structure in neural communication and complex cognitive functions has been well established in the literature [52]. The theory of the shortset path being most efficient has also been questioned. Higher cognitive scores appear negatively correlated with path length but not with clustering or overall connectivity. Patterns of injury based on functional connectivity could provide biomarkers for diagnosis, prognosis and recovery of diseases [29]. Brain networks in adults are not scale free, but instead their degree distributions follow an exponentially truncated power law [48] [1].

While it has been established that electrophysiological methods measure neuronal activity more directly and have better temporal resolution, most studies still use fMRI due to its ubiquity [16]. The small number of subjects and limited data resolution can affect the size of the neural networks and the applicability of inferences from these studies. Preterm infants have been a subject of special interest, but effective EEG studies, differentiating preterm and full-term neuronal networks based on event-related potentials (ERP) data have been missing.

In this study, we hypothesized the differences in small world and community structures of preterm and full-term infants measured using EEG data. We used complex network analysis on time-series tactile stimulus data to show differences in preterm and full-term neuronal networks. We investigated the patterns of connectivity in the community structure and small world networks of term and preterm neural networks. Lastly, we investigated the differences in neural networks from a scale free perspective.

Chapter 2: Background and Related Work

Preterm infants have been at the center of discussions when it comes to neurodevelopmental disorders. Studies have tried drawing parallel between infants, adolescents and adult brains. This section reviews the past studies involving neuronal activity and neurodevelopmental disabilities, explored from a network perspective. Most of these studies involve the use of functional magnetic resonance imaging (fMRI) and utilize various parcellation techniques. A few of these techniques also involve significant amount of human effort.

Schmidt 2007 [39], conducted studies that have examined the short-term stability of resting frontal electroencephalography (EEG) asymmetry measures in typically developing infants. It was found that infants in the Stable Right frontal asymmetry group exhibited a significantly higher baseline heart rate and were rated higher on maternal perception of fear at age 9 months compared with infants in the other two groups, replicating earlier work.

Douw et al. 2011 [21], reported the relation between resting-state functional brain topology with overall and domain-specific cognitive performance in healthy participants and possible gender differences, using magnetoencephalography (MEG) in 14 male and 14 female participants. Better total cognitive performance was related to increased local connectivity in the theta band, higher clustering coefficient (in delta and theta bands) and higher small-worldness (in theta and lower gamma bands). Women showed less clustering and shorter path length in the delta band compared to men, suggesting that women's brains are more efficiently wired than men's brains, which may implicate a functional benefit in females, which is reflected in weaker correlations between network topology and cognitive performance in women.

Fair et al. 2007 [24] calculated the small-world properties of graphs from 39 task control regions from resting-state functional connectivity MRI (rs-fcMRI) data from 210 subjects aged 7-35 years and found that, throughout development from childhood to young adulthood, network clustering coefficient values were near those of lattices, and network path length values were near those of random graphs, indicating that the graphs were small-world networks at all ages examined. This suggested that, despite the large differences in community structure across development, that child, adolescent, and adult networks were all organized in a manner that facilitated simultaneous efficiency on local and global scales.

Power et al. 2010 [46] found that selected short-distance edges tend to be strong in children and weaken over development, in contrast to a subset of long- distance edges, which are typically weak in children and increase in strength over development. Second, these developmental increases in edge strengths tend to occur between nodes that are functionally related in young adults, such as edges between nodes within the default mode network, or edges between nodes of the fronto-parietal task control network. Third, community structure is present and strong in graphs at all stages of development. Fourth, as a result of local decreases and long-distance increases in edge strengths over development, communities in children group nodes largely by anatomical proximity, whereas communities in adults group nodes by functional roles. Finally, despite the reorganization of communities over development, graphs are consistently structured in manners that facilitate efficiency at both local and global network scales.

Power et al. 2011 [45] studied graphs of functional brain organization in healthy adults using resting state functional connectivity MRI. They found that graph measures of the areal network indicate that the default mode subgraphs share network properties with sensory and motor subgraphs: it is internally integrated but isolated from other subgraphs, much like a "processing" system.

Batalle et al. 2012 [10] used diffusion MRI based connectomics to obtain structural brain networks of a prospective cohort of one-year-old infants (32 controls and 24 Intrauterine Growth Restriction (IUGR)) and analyzed the existence of quantifiable brain reorganization of white matter circuitry in IUGR group by means of global and regional graph theory features of brain networks. IUGR infants presented decreased global and local weighted efficiency, and a pattern of altered regional graph theory features. By means of binomial logistic regression, they also demonstrated that connectivity measures were associated with abnormal performance in later neurodevelopmental outcome as measured by Bayley Scale for Infant and Toddler Development, Third edition (BSID-III) at two years of age. They showed that the structural brain networks of 1-year old children are small-world network of neural tracts. Random networks communicate every pair of nodes with relatively few intermediate steps, but have a lack of organization. On the contrary, lattice networks have a high level of organization but a poor average capability to efficiently communicate a given pair of nodes.

Elsabbagh et al. 2010 [22] tried to decipher the answers related to autism (ASD) in babies from the perspective of genetics. They intended to establish a relationship between the autistic babies and their siblings and also wish to identify some early stage biomarkers such as an atypical rapid rate of brain growth. They further related heritability of ASD and the genetic risk contributing to the emergence of the condition.

Jones et al. 2001 [33] compared EEG activity of 10-month-old infants of depressed and non-depressed mothers, during stimuli designed to produce happy and sad responses. Infants of depressed mothers showed greater relative right frontal EEG asymmetry compared to infants of non-depressed mothers, both during baseline and during a play interaction with their mothers. Infants of depressed mothers also showed more negative and less positive expressions. They further relate the frontal EEG activity to be a biological marker for symptoms of depression, basing their analysis on Multivariate Analysis of Variance (MANOVA).

Vanhatalo et al. 2006 [53] proposed a simple, neurophysiologically and neuroanatomically based framework for neonatal EEG interpretation. This is composed of two developmental trajectories: one related to discrete spontaneous activity transients (SAT) and the other to the ongoing, apparently oscillatory EEG activity.

Liu et al. 2008 [37] investigated the topological properties of human brain functional networks derived from resting-state fMRI. They obtained data from 31 patients with schizophrenia and 31 healthy subjects; then functional connectivity between 90 cortical and sub-cortical regions was estimated by partial correlation analysis and thresholded to construct a set of undirected graphs. Their findings demonstrated that the brain functional networks had efficient small-world properties in the healthy subjects; whereas these properties were disrupted in the patients with schizophrenia. Brain functional networks had efficient small-world properties which support efficient parallel information transfer at a relatively low cost (from Achard and Bullmore, 2007). More importantly, in patients with schizophrenia the small-world topological properties were significantly altered in many brain regions in the prefrontal, parietal and temporal lobes. Patients with schizophrenia showed disturbed topological properties, such as a lower degree of connectivity, a lower strength of connectivity, a lower absolute clustering coefficient and a longer absolute path length compared with those of healthy subjects, indicating a decreased synchronization of functionally related brain regions in schizophrenia.

Chen et al. 2008 [17] investigated the intrinsic modular architecture of the human brain network derived from cortical thickness measurement. They showed that the human cortical network is organized into 6 topological modules that closely overlap known functional domains such as auditory/language, strategic/executive, sensorimotor, visual, and mnemonic processing, by investigating the modularity of a cortical network consisting of 45 regions and 102 significant connections that was constructed in their previous study using MRI data from 124 normal adults. They also evaluated the topological importance of specific cortical regions and paths, in terms of their contribution to network properties.

Fransson et al. 2011 [26] showed that cortical hubs and their associated cortical networks are largely confined to primary sensory and motor brain regions in the infant brain. Their findings in infants stand in stark contrast to the situation found in adults where the majority of cortical hubs and hub-related networks are located in heteromodal association cortex. They suggest that the functional network architecture in infants is linked to support tasks that are of a perception-action nature. Their results were 3-fold: 1. Explored the potential presence of cortical hubs in the infant brain, 2. Identified the cortical networks associated with cortical hubs in the infant brain, 3. Investigated the overall topographical structure of the functional organization of the infant brain. They had fMRI imaging data from 18 infants. They also concluded that the topological structure of the functional network in the infant brain lies well within the small-world network regime. One of their key finding was that at the time of birth, the functional brain connectome largely involved brain regions responsible for sensation and action, whereas only weak involvement was found for heteromodal brain areas.

Boersma et al. 2011 [14] used graph theoretical concepts to examine changes in functional brain networks during normal development in young children. They used resting-state eyes-closed, 14 channel EEG data from 227 children, recorded twice (once in 5 years, then again in 7 years). Mean normalized clustering index, average path length and weight dispersion were calculated to characterize network organization. For all frequency bands mean synchronization likelihood (SL) decreased from 5 to 7 years. Clustering coefficient increased in the alpha band. Path length increased in all frequency bands. Mean normalized weight dispersion decreased in beta band. Girls showed higher synchronization for all frequency bands and a higher mean clustering in alpha and beta bands. They found an increase in average clustering and path length and decreased weight dispersion indicating that normal brain maturation is characterized by a shift from random to more organized small-world functional networks. Gender effects were found to be significant for the normalized clustering index, showing higher clustering in girls than in boys.

Alexander-Bloch et al. 2010 [4] used graph theory to investigate topology of networks derived from resting-state fMRI data on 13 childhood-onset schizophrenia (COS) patients and 19 healthy volunteers. They measured functional connectivity between each pair of 100 regional nodes, focusing on wavelet correlation in the frequency interval 0.05-0.1 Hz, then applied global and local thresholding rules to construct graphs from each individual association matrix over the full range of possible connection densities. Their major focus in this article was on modularity, using two approaches - k-means unsupervised learning and spin-glass approach - at different resolutions. They found that modularity of brain functional networks was significantly reduced in COS, due to a relatively reduced density of intra-modular connections between neighboring regions, also clustering was decreased, while complementary measures of global efficiency and robustness were increased, in the COS group.

Geary et al. 2002 [27] described the existing studies on brain and cognitive plasticity. They argued that 4 forms of modularity and 3 forms of neural and cognitive plasticity define the relation between genetic constraint and the influence of developmental experience.

Johnson 2001 [31] presented research on the postnatal neuroanatomical development of the human brain, and findings on perceptual and cognitive development during infancy. He described various experiments that have been performed on infants involving tasks like object finding, face recognition and other studies related to perceptual abilities. The three theories of brain development he describes are: 1. The maturational perspective, 2. The interactive specialization approach, 3. The skill-learning hypothesis. He highlighted the need for better non-invasive imaging techniques.

Fair et al. 2009 [23] combined resting state functional connectivity MRI (rs-fcMRI), graph analysis, community detection, and spring-embedding visualization techniques to analyze four separate networks (the cingulo-opercular, fronto-parietal, cerebellar, and default mode networks). They found a trend toward 'segregation' (a general decrease in correlation strength) between regions close in anatomical space and 'integration' (an increased correlation strength) between selected regions distant in space. They stated that this is a general developmental principle for changes in functional connectivity that would extend to large-scale graph theoretic analyses of large-scale brain networks. Communities in children are predominantly arranged by anatomical proximity, while communities in adults predominantly reflect functional relationships. They claimed that the organization of multiple functional networks shifts from a local anatomical emphasis in children to a more "distributed" architecture in young adults. They argued that this local to distributed developmental characterization has important implications for understanding the development of neural systems underlying cognition. They found similar graph theoretical properties, showing a small-world like network in both children and adults, while the community detection (using modularity optimisation) gives stable communities that are different between young children and young adults. They stated that early school age children and adults both have relatively efficient systems that may solve similar information processing problems in divergent ways (implying that communication efficiency in both children and adults is similar). They stated that the regions in close proximity of each other were highly correlated over the development periods. They had a resting-state fcMRI data from 210 subjects, aged 7-31 (66 aged 7-9; 53 aged 10-15; 91 aged 19-31). They also stated that by 8 years old, the networks already display 'small world' properties similar to those of adult networks, indicating that efficient graph structures are already in place for both local and distant processing, though they are organized differently than in later development (which we have found for infants). The size of their graph was limited to only 34 nodes and therefore it is possible that with an increase in number of nodes the specific results identified by them might change. Also, their regions of focus were derived from adult imaging studies.

Fan et al. 2011 [25] analyzed development patterns of brain anatomical networks derived from morphological correlations of brain regional volumes, using longitudinal MRI data of 28 healthy pediatric subjects, collected at their ages of 1 month, 1 year, and 2 years. Their results show that the brain network of 1-month-olds has the characteristically economic small-world topology and nonrandom modular organization. The network's cost efficiency increases with the brain development to 1 year and 2 years, so does the modularity, supporting their hypothesis that the small-world topology and the modular organization of brain networks are established during early brain development to support rapid synchronization and information transfer with minimal rewiring cost, as well as to balance between local processing and global integration of information. They had 3 questions: 1. Do the brain networks have economic small-world topology in early development? 2. Do the brain networks have modular organization in early development? 3. How does the efficiency and modularity of the brain networks change in the course of early development? To draw a parallel between children and adults, they considered MRI images of 27 healthy adult subjects (13 males/ 14 females, 24 ± 3 years). Modularity has been calculated using a method described in (Clauset et al. 2004 [19]). For the statistical significance of graph metrics, they used permutation test. They claimed that the cost efficiency increases with the brain development, similar to the global efficiency. They stated that modularity of the brain networks of two-year-olds is higher than the modularity of the brain networks in younger and adult brains, a similar trend to the local efficiency. They also claimed that this is the first longitudinal study of brain networks with respect to their efficiency and modular organization in early brain development. They admited the limitation of the study being the relatively small sample size. They concluded with the claim that the efficiency and modularity increases with brain development.

Paterson et al. 2006 [44] provided an overview of four domains that have been studied using techniques amenable to elucidating the brain/behaviour interface: language, face processing, object permanence, and joint attention, with particular emphasis on studies focusing on early development. With this review they aimed to stimulate thinking about gaps in our understanding of relationship between brain and behaviour across development. They aimed to provide ideas about candidate brain areas that are likely to be implicated in particular behaviours or cognitive domains.

Alexander-Bloch et al. 2012 [3] presented a work based on their previous paper (2010, given above), where they showed that modularity decreased in functional brain networks in schizophrenia. They introduced a method based on the normalized mutual information between pairs of modular networks to show that the community structure of the brain network is significantly altered in schizophrenia (concomitant with a decrease in modularity), using resting-state fMRI in 19 participants with childhood-onset schizophrenia and 20 healthy participants. They found that modularity is decreased in the functional networks of patients with childhood-onset schizophrenia, implying that there are relatively more connections between modules, and fewer connections within modules, in the patient population, which in-turn implies that the anatomical identity of the brain regions comprising specific functional modules is altered in schizophrenia. They found among the most consistent modules are the occipital module, the subcortical module, and at least in the healthy participants the primary motor/somatosensory module. They also showed that within-group similarity in the brain functional community organization is significantly higher than the between-group similarity.

van den Heuvel et al. 2013 [52] discussed network based approaches, the brain hubs and their roles in information integration underpinning numerous aspects of complex cognitive function. Their goal is was to examine the concept of network hubs in the context of brain data, with respect to their central placement in the overall network structure and their putative role in neural communication and integrative brain function. They discussed recent findings of network studies that highlight the central role of candidate hubs in both the healthy and diseased brain, offering a conceptual framework that examines potential functional roles of neural hubs from the perspective of network science, especially in the context of network models of communication, integration, and information flow. Their major focus has been on fMRI data and how Pearson cross-correlation helps to define brain networks. They had an anatomical perspective, describing the ‘important’ brain regions and drawing a parallel between the functional and structural similarities of human brain with that of cat and macaque and argue that the structural hubs remain the same despite different data collection methodologies. They also attributed fictional connectivity of child brain to heredity. They raised an important question demeaning the assumption that shortest paths in the brain network are most efficient and stated that we do not have enough data on these special subjects that traces network paths for the flow of information. Maybe answering this question requires a lot more data and detailed scrutiny. They say that the imbalance between incoming and outgoing projections suggests a potential role for cortical hub regions as neural communication ‘sources’ and ‘sinks’ and also that we cannot fully predict the dynamic patterns of communication. They stated that hub models of brain connectivity suggest that the short paths contributing to small-world are not randomly placed within the network architecture, but these shortcuts rather aggregate at the hub nodes. They concluded with the following questions: 1. What is the most sensitive and reliable way to detect network hubs? 2. To what extent can structural network models of the brain predict the location of functional hubs? 3. Do hub nodes differ from non-hubs in their gene-expression and metabolic profiles? 4. What are the developmental mechanisms through which hub regions emerge? 5. To what extent do hubs form global sources and sinks of neural activity? 6. Which brain and mental disorders can be understood as ‘disorders of brain network communication’? 7. Are brain hubs potential ‘hot spots’ for developing new diagnostic biomarkers or attractive targets for therapeutic intervention?

Hagmann et al. 2010 [28] explored the contribution of white matter maturation to the development of connectivity for 30 subjects between ages 2 and 18 years using high b-value diffusion MRI tractography and connectivity analysis. They measured changes in connection efficacy as the inverse of the average diffusivity along a fiber tract and observed significant refinement in specific metrics of network topology, including a significant increase in node strength and efficiency along with a decrease in clustering. Major structural modules and hubs were in place by 2 years of age, and they continued to strengthen their profile during subsequent development. They also recorded resting state fMRI from a subset of subjects confirming a positive correlation between structural and functional connectivity, and in addition observed that this relationship strengthened with age. They found that network refinement mediated by white matter maturation promotes increased global efficiency. They hypothesized that the functional phenomena are driven through structural network refinement, such as regionally specific increases in myelination, axonal diameter, remodeling of dendritic arborization, and neurochemical changes. They used cortical parcellation in addition to using apparent diffusion coefficient (ADC) as a marker of white matter maturation. They found that brain structural networks exhibit robust small-world properties across the entire range of development examined. The small-world index generally decreases with age, with significant negative correlations found for individual cortical hemispheres and whole-brain networks, at both low and high resolutions. Modularity did not exhibit consistent significant trends with age, while centrality remained largely unchanged and structural clustering coefficient exhibited a marked decrease with age. Overall, they wanted to test how the underlying structural substrate is reshaped by white matter maturation during this important developmental period and how these structural modifications parallel these established functional trends. They suggested that maturation of interregional pathways results in topological changes in structural brain networks.

Micheloyannis et al. 2006 [40] characterized the spatial pattern of functional connectivity by computing the synchronization likelihood (SL) of EEG (from a 28 cap electrode) at rest and during performance of a working memory task using letters of the alphabet presented on a PC screen in subjects with schizophrenia (20 subjects, 15 male, 5 female, mean age 32.4 years, mean duration of illness 10 years) and 20 healthy controls. They tested if a disruption of an optimal spatial pattern ("small-world") of the functional connectivity network underlies schizophrenia. During the working memory (WM) task healthy subjects exhibited small-world properties, in alpha, beta and gamma bands, which were not exhibited by the schizophrenic group. They claimed their findings to be in accordance with a partially inadequate organization of neuronal networks in subjects with schizophrenia. Coherence is not sensitive to nonlinear dynamical interdependences (Break- spear et al., 2003), which is why they came up with SL method. The schizophrenia group showed a significantly lower Clustering Coefficient at rest and during WM, while the path length was significantly higher at rest, but not during WM. The findings of small-world network disturbances in schizophrenia are indicative of a partial disorganization of neural networks in this illness. They claimed that their work is the first such study in schizophrenia research.

Bullmore et al. 2009 [16] presented a very broad overview of the field and studies, reviewed different methods and role of networks with bits and pieces of anatomical review. They reviewed studies investigating complex brain networks in diverse experimental modalities (including structural and functional MRI, diffusion tensor imaging, MEG and EEG in humans) and provided an elementary introduction to the basic principles of graph theory. They discussed clustering of functionally related areas with short average path lengths, which are indicative of the small-world architecture. They also provided an anatomical perspective where they stated that the precuneus, the insula, the superior parietal cortex and the superior frontal cortex show high ‘betweenness centrality’ and thus constitute putative hubs. Using parcellation, they found highly clustered networks, with majority of paths existing in close anatomical proximity. The precuneus is involved in self-referential processing, imagery and memory, and its deactivation is associated with anaesthetic-induced loss of consciousness. Mapping functional networks using fMRI, they found that high-degree nodes or hubs of this network were mostly regions of multimodal association cortex, and the degree distribution was described by an exponentially truncated power law. The older age group also showed evidence of small-world properties, but had significantly reduced cost efficiency as they had to be relatively over-connected in order to provide efficiency for parallel information transfer as seen in younger subjects. They stated that the electrophysiological methods measure neuronal activity more directly and have better temporal resolution. Many large networks exhibit scale free power law degree distributions, indicative of existence of highly connected nodes. It is notable that pure power law scaling of the degree distributions of human brain functional networks has only been reported by voxel-level analysis, whereas exponentially truncated power laws have been reported by region-level analysis. Their findings indicate that brain's structural and functional networks are intimately related and share common topological features, such as modules and hubs. They stated that the brain networks derived from fMRI, EEG or structural MRI data are altered in patients with schizophrenia or alzheimer's disease (as seen from papers cited above) and loss of small-world network properties might provide a clinically useful diagnostic marker. They quoted 3 case studies, showing how networks confirmed a degradation of small-world attributes in patients with Alzheimer's disease and suggested that this effect is due to disease-related changes at highly connected network hubs. They also discussed high heritability in case of small world networks from EEG and indicated the importance of short path length in the archetypal brain.

Sporns 2010 [48] provided an overall idea of the field, discussed the relevance of theoretical and methodological foundations of network theory and linked them with possible applications in the field. High clustering and a short path length are the defining characteristics of a universal class of network architectures found in social, technological, and biological systems, including the brain (Sporns and Zwi, 2004). The modular small-world networks encountered in the brain not only allow for efficient information processing but are economical with respect to their wiring and metabolic cost (Bassett and Bullmore, 2006). He discussed the criticalness of the centrality and its influence for predicting functional disturbances that occur upon node or edge deletion, highlighting the importance of scale-free architecture of networks (if you attack the hubs, the system can break down with a high probability).

Hart et al. 2016 [29] described the ideas and concepts behind the connectome and its analysis with graph theory and how to form a connectome using resting state functional MRI data as an example. They also highlight selected insights into healthy brain function that have been derived from connectome analysis and illustrate how studies into normal development, cognitive function, and the effects of synthetic lesioning can be relevant to neurosurgery. They conclude by providing a pr é cis of early applications of the connectome and related techniques to traumatic brain injury, functional neurosurgery, and neurooncology. Experiments on intelligence and network found that a higher IQ was negatively correlated with path length but not with clustering or overall connectivity. They stated that the patterns of injury based on functional connectivity could provide biomarkers for diagnosis, prognosis and recovery. In mild Traumatic Brain Injury (TBI) patients, networks had a longer average path length, reduced overall cost, and reduced network efficiency compared with controls.

Achard et al. 2007 [2] investigated the efficiency and cost of human brain functional networks measured using fMRI in a factorial design: two groups of healthy old (N = 11; mean age = 66.5 years) and healthy young (N = 15; mean age = 24.7 years) volunteers were each scanned twice in a no-task or "resting" state following placebo or a single dose of a dopamine receptor antagonist (sulphide 400 mg). Functional connectivity between 90 cortical and subcortical regions was estimated by wavelet correlation analysis and thresholded to construct undirected graphs. These brain functional networks were small-world and economical in the sense of providing high global and local efficiency of parallel information processing for low connection cost. Efficiency was reduced disproportionately to cost in older people, and the detrimental effects of age on efficiency were localized to frontal and temporal cortical and subcortical regions. Dopamine impaired local and global efficiency but it did not interact with the effect of age. They claim that brain functional networks have economical small-world properties — supporting efficient parallel information transfer at relatively low cost — which are differently impaired by normal aging and pharmacological blockade of dopamine transmission. The paper revolves around the main hypothesis that small-world brain functional networks have economical properties of high global and local efficiency for low cost. They found that efficiency monotonically increased as a function of cost in all networks; the random graph had higher global efficiency than the lattice; and the lattice had higher local efficiency than the random graph. Also, older people had reduced global and local efficiency of brain functional networks compared with younger people. Complete sample: 30 healthy human volunteers were recruited in two age groups: 17 younger participants aged 18-33 years, mean age = 24.3 years, nine male; and 13 older participants aged 62-76 years, mean age = 67.3 years, six male.

Given the large corpus of literature and increased effectiveness of EEG data for recording brain activity, studies involving term and preterm infants, investigating the differences in neural networks of the two was still missing from the field. Also, most of these studies involve the use of fMRI data instead of more effective EEG.

Chapter 3: Methodology

The data used in this study were a subset from a prospective observational cohort of infants cared for in the newborn and intensive care nurseries at Vanderbilt University Medical Center between May 2013 and May 2014. We included full-term infants (38-42 weeks estimated gestational age (EGA)) and preterm infants (24-30 weeks EGA) cared for in the Neonatal Intensive Care Unit (NICU) for a minimum of 48 hours. We excluded any full-term infants with documented maternal opiate use within 48 hours of testing time or concerns for intrauterine drug exposures from the medical team, any preterm infants receiving opiates or sedatives within 48 hours of testing time or those with any antiepileptic drug use since birth. We also excluded infants with lethal congenital abnormalities or severe abnormalities on any cranial imaging (cerebellar hemorrhage, intraventricular hemorrhage grade III or IV, periventricular leukomacia, ischemia or stroke) or infants who had culture-proven sepsis, meningitis or necrotizing enterocolitis Stage IIA or greater, or severe BPD. Parents consented prior to testing using Vanderbilt IRB-approved protocols.

3.1 Somatosensory Testing

Continuous EEG data were acquired using published protocols as near to discharge as possible in preterm infants (35-38 weeks Postmenstrual age (PMA)) and between 1 and 3 days after birth for full-term infants. Briefly, a high-density array net of 128 electrodes embedded in soft sponges (Geodesic Sensor Net, EGI, Inc., Eugene, OR [32]) recorded the EEG using NetStation software (v. 4.3; EGI, Inc., Eugene, OR). Data were sampled at 1000Hz. All infants were tested in his/her patient room while lying on their backs in the bassinet/crib or being held in the supine position by a caregiver. No restraint was used, and infants were tested in a quiet alert or drowsy state. Tactile stimulation approximating light touch was delivered by means of air puffs emanating from a nozzle positioned 5 mm above the skin of the palmar surface of the right hand using a mold holder. The puff delivers a calibrated pressure of $5psi$ over a $3mm^2$ area. Over a 5-minute experimental session, 60 puff stimuli were delivered at random inter-trial intervals, with a minimum of 2500 ms between puffs and no greater than 2 consecutive puffs to prevent habitation. ERP data were pre-processed according to published protocols and using NetStation algorithms.

3.2 Generation of mean amplitude components

Based on our prior studies [38], we identified a time window in term and preterm infants during which the global field power (a reference-independent measure of the electric field of the brain across the entire scalp) [36] [42] was most significantly different from a sham control stimulus (171-240 ms post-stimulus). This served as a time window in which to explore functional connectivity in processing of light touch. Baseline EEG during the 200ms prior to the stimulus served as a time window to study resting state connectivity. For each electrode, the mean amplitude was calculated across the time interval.

3.3 Network Analysis

3.3.1 Community Detection using modularity maximization

Modularity is a scale value (ranging between -1 and 1) that relates the structure of the community with the network outside the community by comparing the density of the edges within to those present outside the community. The *Louvain Method* is a greedy optimization method (complexity $O(nlogn)$), iteratively optimizing local communities until global modularity can no longer be improved, given small perturbations in community state [13].This method is an unsupervised, two-phase iterative method.

In phase 1 of the algorithm, we started with a network of N nodes, assigning a separate community to each node, so that the total communities, at this step, are equal to the number of nodes in the network at the given resolution. For each node i, we considered its neighbor j. Node i, was then removed from its own community and placed into the community of j. We then calculated the gain in modularity (equation given) for the community from which i was removed and for the neighbor j's community, to which i was added. We then placed i in j's community, the j for which the gain (positive) is maximum. For a tie in any two communities, we used a random tie-breaking rule. In case if there was no positive gain possible, we left i in its original community.

$$\Delta Q = \left[\frac{\Sigma_{in} + 2k_{i,in}}{2m} - \left(\frac{\Sigma_{tot} + k_i}{2m}\right)^2\right] - \left[\frac{\Sigma_{in}}{2m} - \left(\frac{\Sigma_{tot}}{2m}\right)^2 - \left(\frac{k_i}{2m}\right)^2\right]$$

where, Σ_{in}: Sum of weights of edges inside the community

Σ_{tot}: sum of all the weights of links to nodes in the community

k_i: Sum of weights of edges connected to i (can be thought of as weighted degree of i)

$k_{i,in}$: Sum of weights of all the edges from i to nodes inside the community

m: Sum of all edge weights in graph G (since we have an unweighted graph, consider weight to be 1, so m here is the number of edges in the graph)

In phase 2 of the algorithm, the communities found in the first phase were then considered as individual nodes. Weights of links between the new nodes were given by sum of weights of links between nodes in the two communities in consideration. (Weight between the two nodes (two communities actually) was given by the sum of weights between all the nodes in the two communities (which is referred as the new node)). Links between the nodes of the same community (referred as the new node in this phase) lead to self-loops for this community in the new network.

$$Q = \frac{1}{2m} \Sigma_{ij} \left[A_{ij} - \frac{k_i k_j}{2m} \right] \delta(c_i, c_j),$$

where, $A_{i,j}$: Edge weight between nodes i and j

k_i: Sum of weights of edges connected to i (can be thought of as weighted degree of i)

m: Sum of all edge weights in graph G (since we have an unweighted graph, consider weight to be 1, so m here is the number of edges in the graph)

c_i: community to which node i is assigned

δ : Simple $\delta - function$ (delta function), equals 1 if all the parameters are equal, else equals 0

Both phases were iterated until modularity (equation given) was maximized. The order in which the nodes were considered may have affected computation time but not modularity. The output of this analysis generates community graphs identifying nodes in each community, a modularity coefficient for the entire network and absolute degree/degree distribution for all nodes within communities.

3.3.2 Small World Detection

We started with a graph G of N nodes and checked if the graph is connected or if it is disjointed, i.e. two or more separate subgraphs existing in the same network without a common edge or if there were any free nodes in the network, which were neither connected together nor part of any subgraph. All the free nodes were removed from the network. The disjointed subgraphs of a particular network were filtered and treated as separate networks. If the graph was connected, with no disjointed components, we left it as it is. We then proceeded with the following algorithm for a test of small world [30].

We defined, L_g: mean shortest path length of graph G

C_g^Δ: Clustering coefficient of G

L_{rand}: mean shortest path of random $E-R$ graph

C_{rand}^Δ: Clustering coefficient of random $E-R$ graph

where the definition of clustering coefficient, based on transitivity, was given by:

$$C^\Delta = \frac{3 \times \text{number of triangles}}{\text{number of paths of length 2}}$$

with triangle being the set of three nodes, with each node connected to the other two.

A network in the analysis was considered small world if $L_g > L_{rand}$ and $C_g^\Delta \gg C_{rand}^\Delta$.

For a given graph G, 3 corresponding $E-R$ random graphs were generated, with the same number of nodes as that of G, but with three different edge probabilities, namely 0.3, 0.5 and 0.7.We then calculated the clustering coefficients and shortest path lengths for both G as well as the three randomly generated $E-R$ graphs and calculated the small world score, as a testing measure for small world, and the ratio of clustering coefficient of G to that of the random $E-R$ graph,

$$\gamma_g^\Delta = \frac{C_g^\Delta}{C_{rand}^\Delta}$$

We then calculated the ratio of shortest path length of graph G to that of random E-R graph,

$$\lambda_g = \frac{L_g}{L_{rand}}$$

Small-world-ness, S^Δ (Small-World score) was then calculated as

$$S^\Delta = \frac{\gamma_g^\Delta}{\lambda_g}$$

A network was a small world, if $S^\Delta > 1$. The test was then repeated with the random E-R graphs with different edge probabilities, keeping the original graph, the one being tested for small-world-ness, intact.

In case of joint graphs or for separated subgraphs of considerably large size, where the Small-World score was not greater than 1 ($S^\Delta \ngtr 1$), we fragmented the network into various smaller connected components which could be small worlds. Breaking the network into smaller connected component subgraphs was accomplished using *NetworkX's* subroutine called *connected_ component_ subgraphs*. First, we iterated over all the vertices v in the given Graph G, checking for the vertices which had not been visited. If the vertex v had not been visited, we computed the single source shortest path length of v in G. All vertices were visited, returning the list of single source shortest path length for v and parsing it to so that for each connected component c, we created a subgraph, given the list returned from part 1 for c.

Single source shortest path length was defined as the shortest path length from source (here, vertex v) to all other reachable nodes in the Graph G. If the node was not reachable from the source node, it was automatically treated as a separate component).

3.3.3 Scale-Free Testing

Any network in general is considered to be a scale-free network if its *degree distribution* follows *power law*, at least asymptotically. If the clustering coefficients follow power law as well, the network is scale free. A non scale-free small world has a lower degree distribution and high clustering coefficient compared to a scale free small world. For our given data, we performed hypothesis testing, with the null-hypothesis being that the given data follows power law, based on the methods described by Clauset et al., 2009 [20] and using *python's* package *power law* [5] . In addition to hypothesis testing based on *p-value*, the data were also subjected to testing based on the *Loglikelihood ratio*, where the other distribution in consideration was *exponential*.

3.3.4 Comparison of Common v/s Uncommon Nodes

We considered the list of the node set for the smaller connected component networks used in the small world analysis. The various smaller connected component networks were identified manually from the average of all the trials for a particular infant and stored separately. For each infant, the node set of the connected components was considered, in the given range and referenced against the grand average network of all full-term infants in the predefined time ranges. The nodes for the manually identified networks for a particular range were considered to be one set while the nodes for the grand average of the corresponding range and type were considered to be the other. We determined the nodes common to both the sets using set intersection. Similarly, uncommon nodes were determined by the operation of set difference over the two sets in consideration. Iteratively repeating the process for the average of trials for every infant and storing the number of nodes occurring in the common and uncommon sets, we plotted the number of common nodes on the abscissa ($x - axis$) against the number of uncommon nodes on the ordinate ($y - axis$).

Chapter 4: Results

Eighty-six infants, 54 full-term and 32 preterm, with estimated gestational age (EGA) 38-42 and 24-30 weeks, respectively, underwent EEG recordings at the Vanderbilt University Medical Center. Post removal of channels not associated with neural vectors (ear/eyebrow), the following results were obtained using our methodology.

4.1 Community Analysis

For the time windows in consideration, namely -200 to 0 (baseline), 171 to 240 (response to stimulus) we observed similar network graphic representations, across full-term and preterm groups at baseline. For response networks, graph patterns appeared different from baseline, but displayed similar patterns between full-term and preterm groups. Community networks graphs and their corresponding trace matrices were similar between full-term and preterm groups. The size of communities at baseline was similar, with the three largest communities in the full term group consisting of 19, 14 and 13 nodes, and in the preterm group consisting of 17, 15 and 11 nodes. During the tactile response, networks appeared denser, with an increased number of edges but a constant number of vertices across the entire network. To illustrate this, the number of vertices in the largest community in the full-term group's response network consisted of 74 nodes, but only 45 nodes in the preterm

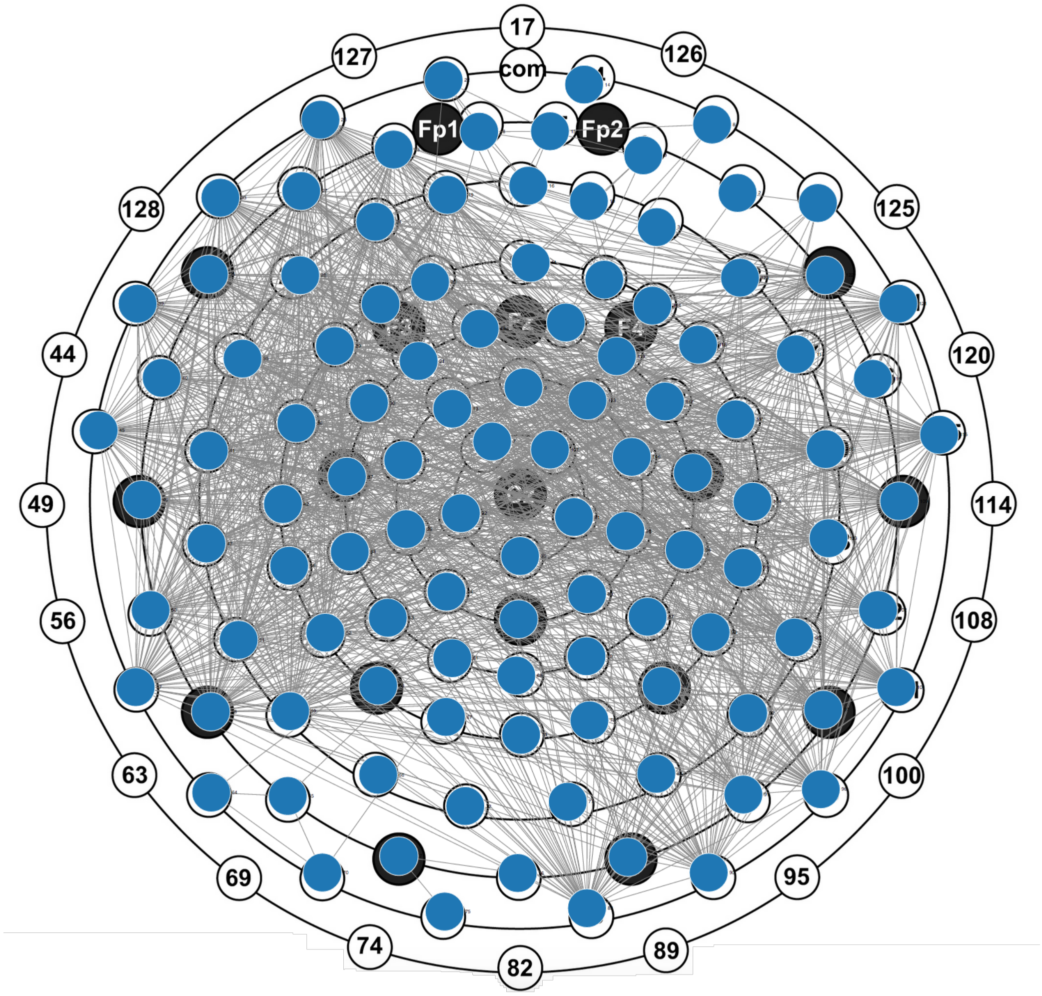


Figure 4.1: Grand Average Network for Preterm in Tactile (Time Window of 171-240 ms after the stimulus)

group, with less densely connected network. The largest three communities, along with their nodes and degreesfor baseline and response networks are shown in Table 4.2.

Across all infants, modularity scores are higher at baseline than during the tactile response (0.551 StdDev. 0.17 vs 0.434 StdDev. 0.20, p <0.001) indicating that this community network organization is less robust during tactile signal processing. For baseline networks, increasing modularity was associated with increasing gestational age at birth (R(85) =0.254, p =0.03). This was not the case for tactile networks. (Table 4.1)

In Figure 4.2, Graphs represent community partitions, color coded into different communities, using the Louvain method. Vertices of the same color belong to the same community. The adjacency matrix of the corresponding graphs is pictured on the right, with colored pixels indicating an edge between the vertices at corresponding row and column. The vertices enclosed in blue squares, running along the primary diagonal of the matrix, are members of individual communities represented in the graph to the left.

Time Window	**full-term**	**Preterm**
-200 to 0 (baseline)	0.619	0.689
171 to 240 (tactile)	0.020	0.093
341 to 499 (late response)	0.440	0.340

Table 4.1: Modularity table for Grand Average Networks

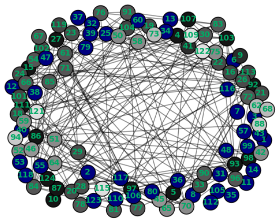
Time Window	Full Term			Preterm		
	Community Number	Number of Nodes	Node:Degree	Community Number	Number of Nodes	Node:Degree
-200 to 0 (baseline)	35	19	{96: 13, 97: 6, 98: 12, 101: 9, 102: 6, 103: 2, 104: 5, 109: 3, 79: 6, 116: 4, 85: 10, 86: 12, 87: 5, 88: 1, 90: 11, 91: 14, 92: 14, 93: 7, 94: 3}	15	17	{96: 5, 66: 7, 67: 3, 71: 6, 72: 6, 76: 9, 77: 9, 78: 2, 81: 4, 83: 8, 84: 10, 85: 10, 54: 1, 90: 2, 91:,7, 61: 7, 30: 2}
	10	14	{34: 11, 35: 14, 36: 9, 38: 10, 40: 11, 42: 8, 12: 11, 45: 8, 46: 12, 50: 1, 67: 4, 59: 1, 29: 12, 30: 4}	3	15	{97: 13, 98: 7, 4: 5, 5: 6, 103: 11, 124: 1, 10: 1, 107: 5, 15: 3, 86: 11, 87: 5, 111: 2, 27: 8, 92: 12, 93: 11}

	32	13	{64: 3, 65: 5, 66: 2, 115: 3, 70: 5, 71: 6, 72: 5, 75: 7, 76: 8, 77: 13, 78: 13, 83: 9, 84: 8}	13	11	{33: 5, 18: 6, 19: 5, 20: 5, 21: 2, 22: 6, 23: 6, 24: 8, 26: 9, 28: 3, 29: 3}
171 to 240 (tactile)	3	74	{4: 82, 5: 81, 6: 81, 7: 79, 11: 79, 12: 81, 30: 82, 35: 82, 36: 82, 37: 82, 38: 82, 39: 82, 40: 82, 41:,82, 42: 82, 45: 82, 46: 82, 50: 82, 51: 82, 52: 81, 53: 82, 54: 82, 57: 82, 58: 82, 59: 82, 60: 82, 61: 82, 62: 82, 64: 82, 65: 82, 66: 82, 67: 82, 70:,82, 71: 82, 72: 82, 73: 82, 75: 82, 76: 82, 77: 82, 78: 82, 79: 82, 80: 70, 81: 82, 83: 82, 84: 82, 85: 82, 86: 82, 87: 82, 88: 82, 90: 82, 91: 82, 92:,82, 93: 82, 94: 82, 96: 82, 97: 82, 98: 82, 99: 82, 101: 82, 102: 82, 103:,82, 104: 82, 107: 82, 109: 82, 110: 82, 111: 82, 112: 82, 113: 82, 115: 82, 116: 82, 117: 82, 119: 82, 121: 82, 122: 82}	3	45	{6: 46, 27: 52, 28: 52, 29: 52, 30: 36, 31: 1, 33: 52, 34: 52, 35: 52, 36: 52, 37: 49, 38: 52, 39: 52, 40:,52, 41: 52, 42: 52, 43: 52, 45: 52, 46: 52, 47: 52, 50: 52, 51: 52, 52: 52, 54: 4, 57: 52, 58: 54, 59: 52, 88: 43, 94: 50, 96: 52, 97: 52, 98: 52, 99:,50, 101: 52, 102: 53, 103: 49, 107: 44, 109: 52, 110: 52, 111: 52, 113: 49, 115: 53, 117: 52, 119: 48, 121: 52}

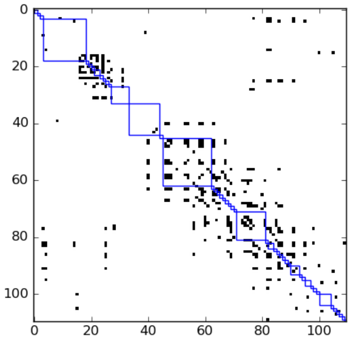
	7	14	{32: 10, 33: 82, 34: 83, 68:,82, 43: 12, 47: 81, 48: 83, 55: 79, 25: 3, 26: 6, 27: 4, 28: 83, 29: 82, 31:,82}	0	20	{1: 7, 2: 2, 123: 16, 7: 2, 104: 10, 105: 14, 83: 54, 12: 51, 77: 2, 93: 3, 19: 44, 20: 52, 122: 53, 23:,46, 24: 52, 84: 19, 26: 52, 91: 54, 60: 4, 90: 54}
	2	7	{3: 2, 9: 4, 10: 1, 15: 3, 18:,3, 19: 5, 105: 2}	2	15	{112: 1, 4: 3, 5: 2, 8: 3, 9:,5, 10: 3, 11: 2, 13: 1, 15: 6, 16: 6, 18: 6, 21: 8, 22: 5, 92: 2, 106: 3}
341 to 499 (late response)	0	31	{1: 26, 2: 26, 3: 12, 4: 24, 5:,28, 6: 27, 7: 2, 11: 11, 12: 23, 20: 24, 27: 25, 31: 16, 54: 27, 61: 31, 67:,32, 72: 32, 104: 25, 105: 24, 106: 5, 109: 23, 110: 25, 111: 26, 112: 28, 113: 15, 116: 26, 117: 26, 118: 27, 119: 3, 121: 24, 122: 27, 123: 28}	4	22	{5: 2, 7: 12, 51: 3, 52: 4, 57:,6, 58: 9, 59: 27, 60: 1, 64: 25, 66: 20, 67: 11, 68: 15, 70: 24, 71: 24, 72:,14, 73: 12, 75: 21, 76: 15, 78: 2, 81: 11, 83: 15, 92: 11}
	17	22	{53: 23, 58: 14, 60: 14, 64:,14, 65: 17, 66: 17, 68: 1, 70: 17, 71: 18, 73: 3, 75: 17, 76: 17, 77: 15, 78:,13, 79: 1, 81: 2, 83: 17, 84: 15, 85: 1, 88: 6, 90: 15, 94: 1}	1	15	{65: 20, 2: 4, 6: 18, 109: 6, 110: 8, 111: 11, 115: 11, 53: 9, 118: 15, 121: 18, 122: 16, 123: 17, 61: 8, 62: 1, 117: 11}

	1	17	{33: 1, 34: 7, 35: 6, 36: 3, 8:,6, 9: 5, 10: 4, 80: 6, 40: 4, 18: 6, 19: 7, 22: 6, 23: 5, 24: 5, 26: 3, 28:,8, 29: 6}	7	6	{10: 1, 18: 4, 19: 4, 22: 5, 23: 10, 24: 14}

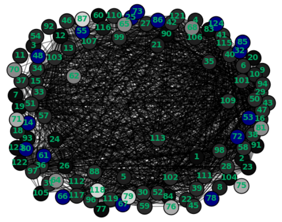
Table 4.2: Nodes in the three major communities found in Grand Average community detection for Full Term and Preterm in given time windows



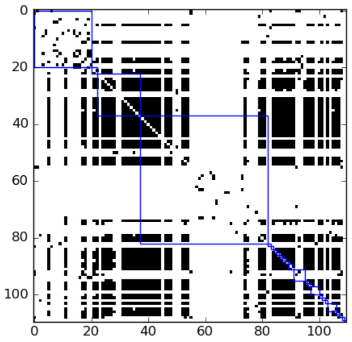
(a) Preterm Baseline (-200 to 0) Community Structure



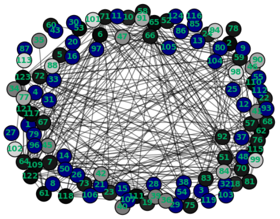
(b) Preterm Baseline (-200 to 0) Adjacency Matrix



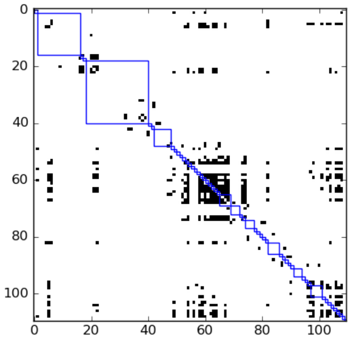
(c) Preterm Tactile (171 to 240) Community Structure



(d) Preterm Tactile (171 to 240) Adjacency Matrix



(e) Preterm Late Response (341 to 499) Community Structure



(f) Preterm Late Response (341 to 499) Adjacency Matrix

(g) Full Term Baseline (-200 to 0) Community Structure

(h) Full Term Baseline (-200 to 0) Adjacency Matrix

(i) Full Term Tactile (171 to 240) Community Structure

(j) Full Term Tactile (171 to 240) Adjacency Matrix

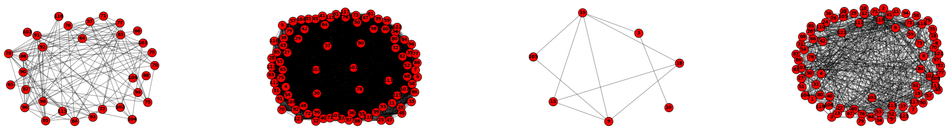
(k) Full Term Late Response (341 to 499) Community Structure

(l) Full Term Late Response (341 to 499) Adjacency Matrix

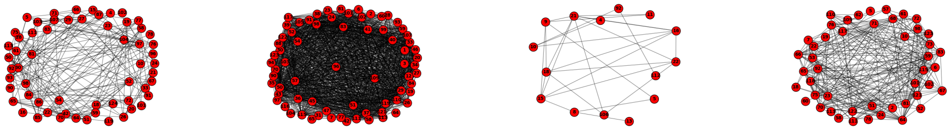
Figure 4.2: Grand Average Communities and corresponding Trace Matrices with communities for Full Term and Preterm in given time windows

4.2 Small World Analysis

Small world connected components at baseline appear less densely connected than during the late response. In the tactile response network, the single small world at baseline appeared to separate into two small worlds, one of very dense and the other sparse (shown in Table 4.3 & 4.4). To compare the component nodes of small worlds, we calculated the percentage of common and uncommon nodes, for each subject and compared to the nodes of the normative grand average small worlds for full term infants in condition. The percentage of common nodes in the smaller of the tactile response small world network was less than in all other conditions. During tactile response, preterm and full-term networks had more similar nodesets than at baseline by Table 4.4.



Small World connected components in full-term infants



Small World connected components in preterm infants

Figure 4.3: Grand average small world representations in full-term and preterm infants for ranges: -200 to 0 prior to stimulus onset (baseline), 171 to 240 (tactile resting state), 171 to 240 (tactile functional), 341 to 499 (late response) respectively

Time Window	**Small World Score (at p=0.3)**	**Average Shortest Path Length**	**Global Clustering Coefficient**	**Number of Common Nodes**
-200 to 0 (baseline)	1.172	2.548	0.585	32
171 to 240 (small)	4.867	1.150	0.977	88
171 to 240 (large)	2.643	1.714	0.580	0

Table 4.3: Table summarising Grand Average Full Term Small World metrics

Time Window	**Small World Score (at p=0.3)**	**Average Shortest Path Length**	**Global Clustering Coefficient**	**Number of Common Nodes**	**Number of Uncommon Nodes**	**% of common nodes**	**% of uncommon nodes**
-200 to 0 (baseline)	0.840	4.145	0.640	26	33	44.1	55.9
171 to 240 (tactile)	2.726	1.789	0.864	62	9	87.3	12.7
171 to 240 (functional)	0.622	2.438	0.424	5	10	33.3	66.7

Table 4.4: Table summarising Grand Average Preterm Small World metrics referenced to full term grand average Small world in corresponding window

4.3 Degree Distributions

The degree distributions for both full term and preterm, across the 2 time windows are shown in Figure 4.4. The full term baseline plot displays a heavy tailed distribution, while the preterm baseline plot does not. The results of tests of power law on degree distribution are documented in Table 4.5 & 4.6 and support that baseline full-term networks do not follow powerlaw (are not scale free) while preterm baseline networks do. During tactile stimulation both full term and preterm small worlds display similar patterns and neither follow power law (small network) nor conform to exponential distributions (large network).

Time Window	**Alpha**	**xmin**	**Loglikelihood Ratio (R)**	**p-value**	**Distribution on the basis of R**	**Distribution on the basis of p-value**
-200 to 0 (baseline)	3.85	7	-2.75	0.008	Exponential	Hypothesis Rejected
171 to 240 (tactile)	2035.97	82	nan	nan	NA	NA
341 to 499 (complex late response)	16.55	26	0.14	0.45	Power	Power

Table 4.5: Comparison/Summary table for Power Law testing of Degree Distribution of full-term Grand Average Networks

Time Window	**Alpha**	**xmin**	**Loglikelihood Ratio (R)**	**p-value**	**Distribution on the basis of R**	**Distribution on the basis of p-value**
-200 to 0 (baseline)	24.61	11	0.17	0.25	Power	Power
171 to 240 (tactile)	193.21	52	nan	nan	NA	NA
341 to 499 (complex late response)	4.74	14	-0.63	0.32	Exponential	Power

Table 4.6: Comparison/Summary table for Power Law testing of Degree Distribution of Preterm Grand Average Networks

(a) full-term Baseline (-200 to 0)

(b) PreTerm Baseline (-200 to 0)

(c) full-term Tactile (171 to 240)

(d) PreTerm Tactile (171 to 240)

(e) full-term Late Response (341 to 499)

(f) PreTerm Late Response (341 to 499)

Figure 4.4: Degree Distribution for Grand Average Networks

4.4 Correlations with Biological Variables

We analyzed changes in Small World scores between baseline and both tactile networks as an indication of organization change upon neural processing of a stimulus. As gestational age at birth increased, Small World score in the large network decreased compared to baseline ($R=0.625$, $p<0.001$) but Small World score in the small network increased ($R= -235$, $p = 0.03$).

Chapter 5: Discussions & Conclusion

This study demonstrated for the first time differences in functional and effective tactile connectivity in full term and preterm infants using complex network analysis. Our baseline network studied the resting state connections of neural signal amplitude as measured using a high-density EEG array, prior to stimulation of the infant hand with a tactile stimulus. At rest, the network exhibits a community structure with modularity increasing with increased maturity at birth. This finding supports analytical work in adults, demonstrating hierarchy of module organization and the economy of brain resources it promotes. It is likely that this modular community structure develops into adulthood. Therefore our study suggests that preterm brains may still be refining this structure. Certainly, work in resting state connectivity using MRI supports evolving network organization in the somatosensory cortex. A novel finding in our study is the observation that modularity of functional resting state networks across term and preterm infants decrease during the tactile stimulus and increase again during the late response.

In tactile window, we found two separate networks of disproportionate sizes. The smaller network is likely to be the functional state network or the functional connectome, involved in the tactile response. The results demonstrate the differences in neurological connectivity between full-term and the pre-term infants in terms of their responses to tactile stimulus. Post stimulus, the neural network tries to eventually regain its resting state topology. This resonates with the presence of transient negative shift, called afterhyperpolarization or the action potential's undershoot phase, post neuronal firing [9] [7]. We observed a smaller functional connectome in full-term tactile windows indicating better efficiency in full-term neural networks for the same task. The second network in tactile window is larger in full-terms than preterms, maintaining a balance with its functional counterpart.

While this study presents for the first time a novel comparison of the functional and effective tactile connectivity in the full-term and pre-term neural networks, considerations about statistical testing methodology merit discussions. Challenges inherent in the methodology include the use of Spearman correlation to build the initial neural networks, instead of Pearson's correlation, given the latter's assumption of data to have a linear relationship, while the earlier being free from any such assumptions of linearity. Spearman correlation is a non-parametric measure, assessing the strength of relationship between the variables, described using a monotonic function. These concerns are not entirely unique to our methodology, but rather general. EEG data is known to follow non-normal distribution [50], which limits the use of pearson correlation.

Another topic requiring careful consideration in building neural networks with such kind of data is choosing an exact threshold or cutoff for number of edges in the network. Thresholding at a fixed number of entities (here edges) is a flawed approach because of its limitations and recalculation with the change in data points or scales. It is possible to consider lesser or non-relevant edges with such an approach. Determining the value of threshold for such a problem is also non-trivial because of limited availability of literature for such a first of its kind study. We used percentage based adaptive thresholding as a solution to this problem, making it problem specific rather than data specific. Considering 5% from both the tails of data for constructing edges in the neural network ensures confinement of both positive as well as negative correlations, while at the same time not being bound by the resolution problems.

In conclusion, we provide the first evidence of differences in the functional and effective tactile connectivity in the neural networks of full-term and preterm infants. We establish the functional connectome involved in stimulus and show that it is different in the two cases. Future work will address a bigger and even corpus of data from full-term and preterm infants, along with analysis of multisensory stimulus. It would address the challenges involved with different diseases among infants and also provide biomarkers for the same. It aims to address the problem of learning disorder with a special focus on specific subjective studies, helping us better understand the two brains and pointing out the problems early in childhood. Another approach in this direction could also be to learn the signals from EEG using deep learning techniques based on convolutional neural network, autoencoders and their other variants. Training deep networks and scaling them to the level of handheld devices would make bedside and offsite predictions of neurodevelopmental disorders possible, benefiting countless infants secluded from healthcare services.

Bibliography

[1] S. Achard, R. Salvador, B. Whitcher, J. Suckling, and E. Bullmore. A resilient, low-frequency, small-world human brain functional network with highly connected association cortical hubs. *J Neurosci*, 26(1):63–72, 2006.

[2] Sophie Achard and Ed Bullmore. Efficiency and cost of economical brain functional networks. *PLoS Comput Biol*, 3(2):e17, 2007.

[3] A. Alexander-Bloch, R. Lambiotte, B. Roberts, J. Giedd, N. Gogtay, and E. Bullmore. The discovery of population differences in network community structure: new methods and applications to brain functional networks in schizophrenia. *Neuroimage*, 59(4):3889–900, 2012.

[4] Aaron F Alexander-Bloch, Nitin Gogtay, David Meunier, Rasmus Birn, Liv Clasen, Francois Lalonde, Rhoshel Lenroot, Jay Giedd, and Edward T Bullmore. Disrupted modularity and local connectivity of brain functional networks in childhood-onset schizophrenia. *Frontiers in systems neuroscience*, 4:147, 2010.

[5] Jeff Alstott, Ed Bullmore, and Dietmar Plenz. powerlaw: a python package for analysis of heavy-tailed distributions. *PloS one*, 9(1):e85777, 2014.

[6] L. A. N. Amaral, A. Scala, M. Barthélémy, and H. E. Stanley. Classes of small-world networks. *Proceedings of the National Academy of Sciences*, 97(21):11149–11152, 2000.

[7] Gianluca Ardolino, B Bossi, S Barbieri, and Alberto Priori. Non-synaptic mechanisms underlie the after-effects of cathodal transcutaneous direct current stimulation of the human brain. *The Journal of physiology*, 568(2):653–663, 2005.

[8] A-L Barabási. Scale-free networks. *Scientific american*, 2003.

[9] Mark W Barnett and Philip M Larkman. The action potential. *Practical neurology*, 7(3):192–197, 2007.

[10] D. Batalle, E. Eixarch, F. Figueras, E. Munoz-Moreno, N. Bargallo, M. Illa, R. Acosta-Rojas, I. Amat-Roldan, and E. Gratacos. Altered small-world topology of structural brain networks in infants with intrauterine growth restriction and its association with later neurodevelopmental outcome. *Neuroimage*, 60(2):1352–66, 2012.

[11] A. T. Bhutta, M. A. Cleves, P. H. Casey, M. M. Cradock, and K. J. Anand. Cognitive and behavioral outcomes of school-aged children who were born preterm: a meta-analysis. *JAMA*, 288(6):728–37, 2002.

[12] H. Blencowe, S. Cousens, D. Chou, M. Oestergaard, L. Say, A. B. Moller, M. Kinney, J. Lawn, and Group Born Too Soon Preterm Birth Action. Born too soon: the global epidemiology of 15 million preterm births. *Reprod Health*, 10 Suppl 1:S2, 2013.

[13] Vincent D Blondel, Jean-Loup Guillaume, Renaud Lambiotte, and Etienne Lefebvre. Fast unfolding of communities in large networks. *Journal of statistical mechanics: theory and experiment*, 2008(10):P10008, 2008.

[14] M. Boersma, D. J. Smit, H. M. de Bie, G. C. Van Baal, D. I. Boomsma, E. J. de Geus, H. A. Delemarre-van de Waal, and C. J. Stam. Network analysis of resting state eeg in the developing young brain: structure comes with maturation. *Hum Brain Mapp*, 32(3):413–25, 2011.

[15] G. M. Buck, M. E. Msall, E. F. Schisterman, N. R. Lyon, and B. T. Rogers. Extreme prematurity and school outcomes. *Paediatr Perinat Epidemiol*, 14(4):324–31, 2000.

[16] E. Bullmore and O. Sporns. Complex brain networks: graph theoretical analysis of structural and functional systems. *Nat Rev Neurosci*, 10(3):186–98, 2009.

[17] Z. J. Chen, Y. He, P. Rosa-Neto, J. Germann, and A. C. Evans. Revealing modular architecture of human brain structural networks by using cortical thickness from mri. *Cereb Cortex*, 18(10):2374–81, 2008.

[18] Krzysztof Choromański, Michał Matuszak, and Jacek Miekisz. Scale-free graph with preferential attachment and evolving internal vertex structure. *Journal of Statistical Physics*, 151(6):1175–1183, 2013.

[19] Aaron Clauset, Mark EJ Newman, and Cristopher Moore. Finding community structure in very large networks. *Physical review E*, 70(6):066111, 2004.

[20] Aaron Clauset, Cosma Rohilla Shalizi, and M. E. J. Newman. Power-law distributions in empirical data. *SIAM Review*, 51(4):661–703, 2009.

[21] L. Douw, M. M. Schoonheim, D. Landi, M. L. van der Meer, J. J. Geurts, J. C. Reijneveld, M. Klein, and C. J. Stam. Cognition is related to resting-state small-world network topology: an magnetoencephalographic study. *Neuroscience*, 175:169–77, 2011.

[22] M. Elsabbagh and M. H. Johnson. Getting answers from babies about autism. *Trends Cogn Sci*, 14(2):81–7, 2010.

[23] D. A. Fair, A. L. Cohen, J. D. Power, N. U. F. Dosenbach, J. A. Church, F. M. Miezin, B. L. Schlaggar, and S. E. Petersen. Functional brain networks develop from a "local to distributed" organization. *Plos Computational Biology*, 5(5), 2009.

[24] D. A. Fair, N. U. Dosenbach, J. A. Church, A. L. Cohen, S. Brahmbhatt, F. M. Miezin, D. M. Barch, M. E. Raichle, S. E. Petersen, and B. L. Schlaggar. Development of distinct control networks through segregation and integration. *Proc Natl Acad Sci U S A*, 104(33):13507–12, 2007.

[25] Y. Fan, F. Shi, J. K. Smith, W. Lin, J. H. Gilmore, and D. Shen. Brain anatomical networks in early human brain development. *Neuroimage*, 54(3):1862–71, 2011.

[26] P. Fransson, U. Aden, M. Blennow, and H. Lagercrantz. The functional architecture of the infant brain as revealed by resting-state fmri. *Cereb Cortex*, 21(1):145–54, 2011.

[27] D. C. Geary and K. J. Huffman. Brain and cognitive evolution: forms of modularity and functions of mind. *Psychol Bull*, 128(5):667–98, 2002.

[28] Patric Hagmann, Olaf Sporns, Neel Madan, Leila Cammoun, Rudolph Pienaar, Van Jay Wedeen, Reto Meuli, J-P Thiran, and PE Grant. White matter maturation reshapes structural connectivity in the late developing human brain. *Proceedings of the National Academy of Sciences*, 107(44):19067–19072, 2010.

[29] M. G. Hart, R. J. Ypma, R. Romero-Garcia, S. J. Price, and J. Suckling. Graph theory analysis of complex brain networks: new concepts in brain mapping applied to neurosurgery. *J Neurosurg*, 124(6):1665–78, 2016.

[30] Mark D Humphries and Kevin Gurney. Network ‘small-world-ness’: a quantitative method for determining canonical network equivalence. *PloS one*, 3(4):e0002051, 2008.

[31] M. H. Johnson. Functional brain development in humans. *Nat Rev Neurosci*, 2(7):475–83, 2001.

[32] Mark H Johnson, Michelle de Haan, Andrew Oliver, Warwick Smith, Haralambos Hatzakis, Leslie A Tucker, and Gergely Csibra. Recording and analyzing high-density event-related potentials with infants using the geodesic sensor net. *Developmental Neuropsychology*, 19(3):295–323, 2001.

[33] Nancy Aaron Jones, T Field, Nathan A Fox, M Davalos, and Charlemagne Gómez. Eeg during different emotions in 10-month-old infants of depressed mothers. *Journal of Reproductive and Infant Psychology*, 19(4):295–312, 2001.

[34] Jongkwang Kim and Thomas Wilhelm. What is a complex graph? *Physica A: Statistical Mechanics and its Applications*, 387(11):2637–2652, 2008.

[35] H. J. Larsson, W. W. Eaton, K. M. Madsen, M. Vestergaard, A. V. Olesen, E. Agerbo, D. Schendel, P. Thorsen, and P. B. Mortensen. Risk factors for autism: perinatal factors, parental psychiatric history, and socioeconomic status. *Am J Epidemiol*, 161(10):916–25; discussion 926–8, 2005.

[36] D Lehmann and W Skrandies. Reference-free identification of components of checkerboard-evoked multichannel potential fields. *Electroencephalography and clinical neurophysiology*, 48(6):609–621, 1980.

[37] Y. Liu, M. Liang, Y. Zhou, Y. He, Y. Hao, M. Song, C. Yu, H. Liu, Z. Liu, and T. Jiang. Disrupted small-world networks in schizophrenia. *Brain*, 131(Pt 4):945–61, 2008.

[38] N. L. Maitre, A. P. Key, O. D. Chorna, J. C. Slaughter, P. J. Matusz, M. T. Wallace, and M. M. Murray. The dual nature of early-life experience on somatosensory processing in the human infant brain. *Curr Biol*, 27(7):1048–1054, 2017.

[39] Jon K. Maner, J. Anthony Richey, Kiara Cromer, Mike Mallott, Carl W. Lejuez, Thomas E. Joiner, and Norman B. Schmidt. Dispositional anxiety and risk-avoidant decision-making. *Personality and Individual Differences*, 42(4):665–675, 3 2007.

[40] Sifis Micheloyannis, Ellie Pachou, Cornelis Jan Stam, Michael Breakspear, Panagiotis Bitsios, Michael Vourkas, Sophia Erimaki, and Michael Zervakis. Small-world networks and disturbed functional connectivity in schizophrenia. *Schizophrenia research*, 87(1):60–66, 2006.

[41] S. P. Miller, V. Ramaswamy, D. Michelson, A. J. Barkovich, B. Holshouser, N. Wycliffe, D. V. Glidden, D. Deming, J. C. Partridge, Y. W. Wu, S. Ashwal, and D. M. Ferriero. Patterns of brain injury in term neonatal encephalopathy. *J Pediatr*, 146(4):453–60, 2005.

[42] Micah M Murray, Denis Brunet, and Christoph M Michel. Topographic erp analyses: a step-by-step tutorial review. *Brain topography*, 20(4):249–264, 2008.

[43] J.-P. Onnela, J. Saramäki, J. Hyvönen, G. Szabó, D. Lazer, K. Kaski, J. Kertész, and A.-L. Barabási. Structure and tie strengths in mobile communication networks. *Proceedings of the National Academy of Sciences*, 104(18):7332–7336, 2007.

[44] Sarah J Paterson, Sabine Heim, Jennifer Thomas Friedman, Naseem Choudhury, and April A Benasich. Development of structure and function in the infant brain: Implications for cognition, language and social behaviour. *Neuroscience & Biobehavioral Reviews*, 30(8):1087–1105, 2006.

[45] J. D. Power, A. L. Cohen, S. M. Nelson, G. S. Wig, K. A. Barnes, J. A. Church, A. C. Vogel, T. O. Laumann, F. M. Miezin, B. L. Schlaggar, and S. E. Petersen. Functional network organization of the human brain. *Neuron*, 72(4):665–78, 2011.

[46] J. D. Power, D. A. Fair, B. L. Schlaggar, and S. E. Petersen. The development of human functional brain networks. *Neuron*, 67(5):735–48, 2010.

[47] Jari Saramäki, Mikko Kivelä, Jukka-Pekka Onnela, Kimmo Kaski, and Janos Kertesz. Generalizations of the clustering coefficient to weighted complex networks. *Physical Review E*, 75(2):027105, 2007.

[48] Olaf Sporns. Networks of the brain: quantitative analysis and modeling. Report, 2010.

[49] K. Supekar, M. Musen, and V. Menon. Development of large-scale functional brain networks in children. *PLoS Biol*, 7(7):e1000157, 2009.

[50] SJ Van Albada and PA Robinson. Transformation of arbitrary distributions to the normal distribution with application to eeg test–retest reliability. *Journal of neuroscience methods*, 161(2):205–211, 2007.

[51] M. P. van den Heuvel, R. C. Mandl, R. S. Kahn, and H. E. Hulshoff Pol. Functionally linked resting-state networks reflect the underlying structural connectivity architecture of the human brain. *Hum Brain Mapp*, 30(10):3127–41, 2009.

[52] M. P. van den Heuvel and O. Sporns. Network hubs in the human brain. *Trends Cogn Sci*, 17(12):683–96, 2013.

[53] S. Vanhatalo and K. Kaila. Development of neonatal eeg activity: from phenomenology to physiology. *Semin Fetal Neonatal Med*, 11(6):471–8, 2006.

[54] Duncan J. Watts and Steven H. Strogatz. Collective dynamics of /'small-world/' networks. *Nature*, 393(6684):440–442, 06 1998.

thmtypetheorem

Appendix A: Basics of Graph Theory

Adopted from: Kierstead H.A. Graph Theory Notes. *School of Mathematical and Statistical Sciences, Arizona State University.* *https://math.la.asu.edu/∼halk/598Fol/(2015)MAT513Lectures.pdf*, 2016

A.1 Introduction

A.1.1 Graphs

Formally a *graph* is an ordered pair $G = (V,E)$ where E is an irreflexive, symmetric, binary relation on V. Since E is symmetric there is no need to keep track of the order of pairs $(x,y) \in E$; since it is irreflexive there are no ordered singletons (x,x) in E. This leads to a more intuitive formulation. We take E to be a set of unordered pairs of elements from V. Elements of $V$are called *vertices*; elements of E are called *edges*. If $x,y \in V$ are vertices and $\{x,y\} \in E$ is an edge we usually (but not always) denote $\{x,y\}$ by the shorthand notation xy. So $xy = \{x,y\} = \{y,x\} = yx$. The vertices x and y are called *ends*, or *endpoints*, of the edge xy. The ends x and y of an edge xy are said to be *adjacent* and the end x is said to be *incident* to the edge xy. We also say that x and y are *joined* (not connected) by the edge xy. Two edges are said to be *adjacent* if they have a common end. In this course, all graphs have a finite number of vertices, unless it is explicitly stated that they have infinitely many. Graphs are illustrated by drawing dots for vertices and connecting adjacent edges by lines or curves.

Our definition of graph is what the text calls a *simple graph*. Most of the time we will only be interested in simple graphs, and so we begin with the simplest definition. When necessary, we will introduce the more complicated notions of *directed graphs* and *multigraphs,* but here is a quick hint. A directed graph $G = (V,E)$ is any binary relation (not necessarily irreflexive or symmetric) on V. In other words E is any set of *ordered* pairs of vertices. If E is a *multiset* then G is called a directed multigraph. If E is a set of subsets of V then G is called a hypergraph.

The study of graph theory involves a huge number of of parameters—see the front and back inside covers of the text. This can be quite daunting. My strategy is to introduce these parameters as they are needed. Please feel free to interrupt lectures to be reminded of their meanings. Most of the time my notation will agree with the text, and I will try to emphasize differences. Next we introduce some very basic notation.

Given a graph G, $V(G)$ denotes the set of vertices of G and $E(G)$ denotes the set of edges of G. Set $|G| := |V(G)|$ and $\|G\| := |E(G)|$. This is not standard, and instead the book uses $v(G) = |V(G)|$ and $e(G) = |E(G)|$. Suppose $v \in V(G)$ is a vertex of G. Define

$$N_G(v) := \{w \in V(G) : vw \in E(G)\}; \qquad E_G(v) := \{e \in E(G) : v \text{ is an end of } e\}.$$

The set $N_G(v)$ is called the (open) *neighborhood* of v, and its elements are called *neighbors* of v. So a vertex w is a neighbor of v iff it is adjacent to v. When there is no confusion with other graphs the subscript G is often dropped. The closed neighborhood of v is $N[v] := N(v) \cup \{v\}$—we dropped the subscript. The set $E_G(v)$ is the set of edges incident to v; again, we may drop the subscript G. The text does not provide notation for this set. For simple graphs $|N(v)| = |E(v)|$. However for multigraphs this may note hold, since two vertices might be joined by several edges. With this in mind, define the degree of a vertex v to be $d_G(v) := |E_G(v)|$, but not that for simple graphs $d_G(v) = |N_G(v)|$.

Finally we mention some simple set theoretic notation. The sets of natural numbers, integers and positive integers are denoted, respectively, by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Z}^+$. For $n \in \mathbb{N}$ set $[n] := \{1, 2, \ldots, n\}$; in particular $[0] = \emptyset$. For a set X and an element y, set $X + y := X \cup \{y\}$ and $X - y := X \smallsetminus \{y\}$.

A.1.2 Proofs by Mathematical Induction

Most proofs in graph theory involve mathematical induction, or at least the Least Element Axiom. Here we quickly review this technique. Also see the discussion in the text on page 72, and especially the *induction trap* on page 107.

[Axiom][Least Element Axiom]Every nonempty set of natural numbers has a least element.

Let S be a set of natural numbers and $B = \mathbb{N} \smallsetminus S$. We would like to prove that $S = \mathbb{N}$ and $B = \emptyset$. Here is a way to organize the proof.

Theorem 1 (Principle of Induction)**.** *Suppose $S \subseteq \mathbb{N}$, and every $n \in \mathbb{N}$ satisfies,*

$$\textit{if } k \in S \textit{ for every natural number } k < n\textit{, then } n \in S. \tag{A.1.1}$$

Then $S = \mathbb{N}$.

Proof. Consider any set $S \subseteq \mathbb{N}$ such that (A.1.1) holds for all $n \in \mathbb{N}$, and let $B = \mathbb{N} \smallsetminus S$. Arguing by contradiction, assume that B is nonempty. Then it has a least element l. Since l is the least element of B, every natural number less than l is in S. Applying (A.1.1) to l yields $l \in S$, a contradiction. □

Using the Principle of Induction to prove that $S = \mathbb{N}$, it suffices to prove (A.1.1) holds for all natural numbers n. Notice that the hypothesis of (A.1.1) always holds for $n = 0$ (why?), and so, if (A.1.1) holds for $n = 0$ then $0 \in S$. But in applications, checking (A.1.1) for $n = 0$ is usually a special case. Here is an example.

Theorem 2. *Every natural number greater than* 1 *has a prime factor.*

Proof. Let $S = \{n \in \mathbb{N} : n \leq 1 \text{ or } n \text{ has a prime factor}\}$. It suffices to show A.1.1. Consider any $n \in \mathbb{N}$ such that $k \in S$ for every natural number less than n. We must show $n \in S$. If $n \leq 1$ then $n \in S$ by definition. So suppose $n \geq 2$. If n is prime then it is a prime factor of itself, and so it is in S. Otherwise, there exist integers a, b such that $1 < a, b < n$ and $ab = n$. Since $a < n$, we have $a \in S$. Since $1 < a$ this means that a has a prime factor p. Since p is a factor of a and a is a factor of n, p is a (prime) factor of n. $\square$

A.1.3 Ramsey's Theorem for Graphs

Ramsey's Theorem is an important generalization of the Pigeonhole Principle. Here we only consider its simplest version applied to graphs.

Let $G = (V, E)$ be a graph, and suppose $X \subseteq V$. The set X is a *clique in* G if $xy \in E$ for all vertices $x, y \in X$. It is an *independent set*, or *coclique*, in G if $xy \notin E$ for all vertices $x, y \in X$. A clique (coclique) X is a b-clique (b-coclique) if $|X| = b$. Let $\omega(G) := \max\{|X| : X \textit{ is a clique in } G\}$*, and* $\alpha(G) := \max\{|X| : X \textit{ is a coclique in } G\}$.

A graph H is a *subgraph* of G, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. It is an *induced subgraph* of G if $H \subseteq G$ and $E(H) = \{xy \in E(G) : x \in V(H) \textit{ and } y \in V(H)\}$. For $X \subseteq V$, $G[X]$ is the induced subgraph of G that has vertex set X. The *complement* of G is the graph, $\overline{G} := (V(G), \overline{E}(G))$, where $\overline{E} := \{xy : xy \notin E \text{ and } x, y \in V(G)\}$.

Theorem 3 (Ramsey's Theorem)**.** *For all graphs G and $a, b \in \mathbb{N}$, if $|G| \geq 2^{a+b-2}$ then $\omega(G) \geq a$ or $\alpha(G) \geq b$.*

Proof. Argue by induction on $n = a+b$. (That is, let S be the set of natural numbers n such that for all positive integers a,b if $n = a+b$, and G is a graph with $|G| \geq 2^{a+b-2}$ then $\omega(G) \geq a$ or $\alpha(G) \geq b$. Show that for all $n \in \mathbb{N}$, if $k \in S$ for all $k \in \mathbb{N}$ with $k < n$ then $n \in S$.) Consider any $n = a+b$ with $a,b \in \mathbb{Z}^+$, and any graph G with $|G| \geq 2^{a+b-2}$.

Base step: $\min\{a,b\} = 1$. Since $|G| \geq 2^{a+b-2} \geq 1$, G has a vertex v. Since $\{v\}$ is both a clique and an independent set, both $\omega(G) \geq 1$ and $\alpha(G) \geq 1$. So we are done regardless of whether $a = 1$ or $b = 1$.

Induction Step: $\min\{a,b\} \geq 2$ (so $a-1, b-1 \in \mathbb{Z}^+$). (We assume the induction hypothesis: the theorem holds for all $a', b' \in \mathbb{Z}^+$ with $a'+b' < a+b$.) Let $v \in V(G)$. Then

$$1 + d_G(v) + d_{\overline{G}}(v) = |G| \geq 2^{a+b-2} = 2^{a+b-3} + 2^{a+b-3}.$$

By the pigeonhole principle, either $d_G(v) \geq 2^{a+b-3}$ or $d_{\overline{G}}(v) \geq 2^{a+b-3}$.

Case 1: $d_G(v) \geq 2^{a+b-3}$. Set $H := G[N_G(v)]$. Then $|H| = d_G(v) \geq 2^{a-1+b-2}$. By the induction hypothesis H contains an $(a-1)$-clique X or a b-coclique Y. In the latter case Y is a b-coclique in G. In the former case $X+v$ is an a-clique in G.

Case 2: $d_{\overline{G}}(v) \geq 2^{a+b-3}$. Set $H := G[N_{\overline{G}}(v)]$. Then $|H| \geq 2^{a+b-1-2}$. By the induction hypothesis H contains an a-clique X or a $(b-1)$-coclique Y. In the former case, X is an a-clique in G. In the latter case $Y+v$ is a b-coclique in G. □

A.1.4 Graph Isomorphism and the Reconstruction Conjecture

In order to study graph theory we need to know when two graphs are, for all practical purposes, the same.

Definition 4. Two graphs G and H are *isomorphic* if there exists a bijection

$$f : V(G) \to V(H) \text{ such that for all } x,y \in V(G), xy \in E(G) \text{ iff } f(x)f(y) \in E(H).$$

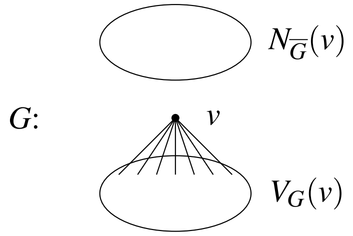


Figure A.1.1: Ramsey's Theorem

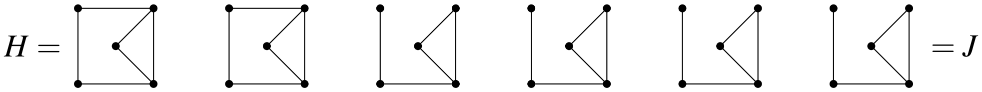


Figure A.1.2: The vertex deleted subgraphs of a graph G. What is $|G|$? What is $\|G\|$? What is the isomorphism type of G?

In this case we say that f is an isomorphism from G to H and write $G \cong H$. The isomorphism relation is an equivalence relation on the class of graphs. The equivalence classes of this relation are called *isomorphism types.* In graph theory we generally do not differentiate between two isomorphic graphs. We say that H is a *copy* of G to mean that $G \cong H$.

If x is a vertex of a graph G then $G - x$ is the induced subgraph $G[V(G) - x]$.

Definition 5. A *multiset of vertex deleted subgraphs* of a graph $G = (V,E)$ is a multiset $\mathscr{G}$ such that there exists a bijection $\psi : V \to \mathscr{G}$ with $\psi(x) \cong G - x$ for all $x \in V$.

Notice that G has infinitely many multisets of vertex deleted subgraphs, but everyone satisfies $|\mathscr{G}| = |G|$; some graphs may appear several times in the multiset $\mathscr{G}$. However, we cannot determine $V(G)$ from $\mathscr{G}$. The following famous conjecture asks whether we can determine the isomorphism type of G from $\mathscr{G}$.

Conjecture 6 (Reconstruction Conjecture)**.** *Let G be an unknown graph with* $|G| \geq 3$. *Given a complete set of vertex deleted subgraphs of G (but not G) it is possible to construct a graph H that is isomorphic to G.*

For $(v,e) \in V \times E$, set

$$\iota(v,e) := \begin{cases} 1 & \text{if } e \in E(v) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } v \in e \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 7 (Handshaking)**.** *Every graph* $G := (V,E)$ *satisfies* $\sum_{v \in V} d(v) = 2\|G\|$. *In particular, G has an even number of vertices with odd degree.*

Proof.

$$\sum_{v \in V} d(v) = \sum_{v \in V} \sum_{e \in E} \iota(v,e) = \sum_{e \in E} \sum_{v \in V} \iota(v,e) = \sum_{e \in E} 2 = 2\|G\|.$$

□

Proposition 8. *For every graph* $G = (V,E)$ *with* $|G| \geq 3$ *and every vertex* $v \in V$,

$$\|G\| = \frac{\sum_{v \in V} \|G - v\|}{|G| - 2} \qquad \text{and} \qquad d_G(v) = \|G\| - \|G - v\|.$$

Proof. Every edge $e \in E$ satisfies $e \in E(G - v)$ if and only if $\iota(v,e) = 0$. Thus

$$\sum_{v \in V} \|G - v\| = \sum_{v \in V} \sum_{e \in E} (1 - \iota(v,e)) = \sum_{e \in E} \sum_{v \in V} (1 - \iota(v,e)) = \sum_{e \in E} (|G| - 2) = \|G\| (|G| - 2).$$

So the first equality holds. The second equality follows from $E = E(G - v) \cup E(v)$. □

Example 9. Suppose $\mathscr{G}$ is the multiset consisting of 2 copies of H and 4 copies of J as shown in Figure A.1.2. Find (with proof) a graph G such that if the multiset of vertex deleted subgraphs of G' is $\mathscr{G}$ then $G' \cong G$.

Solution. Let $G = (V,E)$ be an arbitrary graph for which $\mathscr{G}$ is a multiset of vertex deleted subgraphs. Then $|G| = |\mathscr{G}| = 6$. Using Proposition 8, we have:

$$\|G\| = (2\|H\| + 4\|J\|)/(|G| - 2) = (2 \cdot 6 + 4 \cdot 5)/4 = 8,$$

if $G-x \cong H$ then $d_G(x) = 8 - \|H\| = 2$, and if $G-y \cong J$ then $d_G(y) = 8 - \|J\| = 3$. So G has two vertices with degree 2 and four vertices with degree 3. Choose $x \in V$ with $G-x \cong H$. By inspection, $G-x$ has a unique vertex z with two neighbors w_1, w_2 in $G-x$ such that $d_{G-x}(w_1) = 3 = d_{G-x}(w_2)$. It suffices to show

$$N_G(x) = (V \smallsetminus N_{G-x}[z]) - x :$$

If $G' = (V', E')$ is another graph for which $\mathscr{G}$ is a multiset of vertex deleted subgraphs then there exist distinct $x', z' \in V'$ such that $G' - x' \cong H$, and z' is the unique vertex in $G' - x'$ with two neighbors w_1', w_2' in $G-x$ such that $d_{G'-x'}(w_1') = 3 = d_{G'-x'}(w_2')$. So $G-x \cong H \cong G'-x'$. Let ψ be an isomorphism from G to G'. Then $\psi(z) = z'$, $\psi(N_{G-x}[z]) = N_{G'-x'}[z'])$, and $\psi(N_G(x)) = N_{G'-x'}(x')$. Thus ψ can be extended to an isomorphism from G to G' by mapping x to x'.

Since all $v \in V - x$ satisfy $N_{G-x}(v) \subseteq N_G(v)$, for all $i \in [2]$ we have $3 = d_{G-x}(w_i) \leq d_G(w_i) \leq 3$, and so

$$N_{G-x}(w_i) = N_G(w_i). \tag{A.1.2}$$

Every vertex of $G-x-z$ has degree 2, but J has a unique vertex with degree 1, and removing it leaves another vertex with degree 1. Thus $G-x-z \nsubseteq G-y \cong J$. So $G-z \ncong J$. The only other possibility is that $G-z \cong H$. Thus $2 \leq d_{G-x}(z) \leq d_G(z) = 2$, and so

$$N_{G-x}(z) = N_G(z). \tag{A.1.3}$$

By (A.1.2) and (A.1.3), $N_G(x) \cap N_G[z] = \emptyset$. Since $d_G(x) = 2$, $N_G(x) = (V \smallsetminus N_{G-x}[z]) - x$. See Figure A.1.3. $\square$

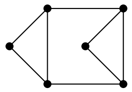


Figure A.1.3: Discovered graph G

A.1.5 Paths, Cycles, Complete Graphs, Petersen's Graph and Decompositions

A *path* is a graph $P = (V,E)$ such that V can be ordered as $v_1,\dots,v_{|P|}$ so that $E = \{v_iv_{i+1} : i \in [|G|-1]\}$. The *length* of the path P is $\|P\|$. Clearly, any two paths with the same length are isomorphic. We use the notation P_n to denote a fixed path of length $n-1$. Then if P is a path of length $n-1$, we say that P is a copy of P_n, or more carelessly $P = P_n$. We write $v_1v_2\dots v_n$ (without commas) to denote a copy of P_n whose edge set is $\{v_iv_{i+1} : i \in [n-1]\}$. Suppose $H \subseteq G$ are graphs. A path $P \subseteq G$ is an H-path if its ends, but not its internal vertices, are contained in $V(H)$. An x,y-path is a path whose first vertex is x and whose last vertex is y.

A *cycle* is a graph C formed by adding the additional edge v_1v_n to a path $v_1v_2\dots v_n$ with $n \geq 3$. Again, the length of C is $\|C\|$. Clearly any two cycles with the same length are isomorphic. We use the notation C_n for a fixed cycle of length n. We write $v_1v_2\dots v_nv_1$ to denote a copy of C_n whose edge set is $\{v_iv_{i\oplus 1} : i \in [n]\}$, where $\oplus$ denotes addition modulo n. The *girth* of a graph G is the length of its shortest cycle C with $C \subseteq G$, if there is one; otherwise the *girth* is infinity.

A *complete graph* is a graph $K = (V,E)$ such that $xy \in E$ for all $x,y \in V$. We use the notation K_n for a fixed complete graph with n vertices. Notice that the vertices of a complete graph are a clique. Then $\overline{K}_n$ is a graph with n vertices and no edges, and the vertices of $\overline{K}_n$ are a coclique. We call $\overline{K}_n$ the *empty graph*. Finally we introduce some notation not in the text:

$K(A,B)$ denotes the graph (V,E) such that $V = A \cup B$ and $E = \{ab : a \neq b \wedge (a,b) \in A \times B\}$. Then $K(A,A)$ denotes a complete graph whose vertex set is A; we abbreviate this by $K(A)$. Finally, for $a,b \in \mathbb{Z}^+$, let $K_{a,b}$ denote a fixed graph of the form $K(A,B)$, where $|A| = a$, $|B| = b$, and $A \cap B = \emptyset$. Such a graph is called a *complete bipartite graph*. (We will have more to say about bipartite graphs shortly.)

The *Petersen* graph has the form $(\binom{[5]}{2}, \{AB : A \cap B = \emptyset, A,B \in \binom{[5]}{2}\})$.

Definition 10. A decomposition of a graph G is a set of subgraphs such that each edge of G appears in exactly one subgraph of the set.

Example 11. K_4 can be decomposed into two P_4's; it can also be decomposed into three P_3's, and into $K_3, K_{1,3}$.

A.1.6 Connection in graphs

Definition 12. A *walk* (in a graph G) is a sequence (list) $W = v_1 v_2 \dots v_n$ of not necessarily distinct vertices such that $v_i v_{i+1}$ is an edge for each $i \in [n-1]$. If the vertices of W are distinct then W is just a path. If $v_1 = v_n$ then W is *closed*; otherwise it is *open*. If W is open then v_1 and v_n are its *ends*, and $v_2, \dots, v_{n-1}$ are its *internal* vertices. If all the edges $v_i v_{i+1}$ are distinct then W is a *trail*. Of course, if the vertices of W are distinct then W is a path. The *length* of W is $n-1$. The walk W is a u,v-walk (trail,path) if $u = v_1$ and $v = v_n$. The trivial walk v_1 of length 0 is closed. We say that W contains a walk W' if there exists a subsequence $W' = v_{i_1} v_{i_2} \dots v_{i_s}$ such that W' is a walk, and each edge $v_{i_h} v_{i_{h+1}}$ of W has the form $v_j v_{j+1}$ for some $j \in [n-1]$. We use the notation Wv_j, v_iWv_j, v_iW to indicate the *subwalks* $v_1 \dots v_j, v_i \dots v_j, v_i \dots v_n$. *Also* $W^* := v_n v_{v-1} \dots v_1$.

Lemma 13. *Every open u,v-walk $W = v_1 \dots v_n$ contains a u,v-path.*

Proof. We argue by induction on the length l of W. If W is a path then we are done. Otherwise, there exist $i, j \in [n]$ with $i < j$ such that $v_i = v_j$. Then $l > 1$ and $W' = v_1 \dots v_i (= v_j) v_{j+1} \dots v_n$ is a shorter u,v-walk contained in W. By the induction hypothesis W' contains a u,v-path P. Clearly, P is also contained in W. □

Definition 14. A graph G is *connected* if for any two distinct vertices u,v there exists a u,v-walk. If there exists a u,v-walk then u is *connected* to v, regardless of whether $uv \in E(G)$. The *connection relation is the set of ordered pairs* (u,v) *of* $V(G)$ *such that there exists a* u,v*-path in G.*

Proposition 15. *The connection relation is an equivalence relation.*

Definition 16. A *component* of G is a subgraph $H = G[X]$ induced by an equivalence class X of the connection relation.

A.1.7 Bipartite graphs

Definition 17. Let $E(A,B)$ denote the set of edges with one end in A and one end in B. A graph $G = (V,E)$ is *bipartite* if it has a *bipartition*, that is a partition of V into one or two independent sets. This means that $E = \emptyset$ or there exists a partition $\{A,B\}$ of V ($V = A \cup B$, $A \cap B = \emptyset$) such that both A and B are independent, or equivalently $E = E(A,B)$. Notice that a graph is bipartite if and only if it is a subgraph of a complete bipartite graph.

Many theorems in graph theory assert the existence of some special structure in a graph—say a bipartition. To show that a particular graph has such a structure it is enough to make a lucky random guess, and check that your guess provides the structure. In general, it is much harder to show that a graph does not have the desired structure. Typically this would require an exhaustive search of exponentially many possibilities—say all $2^{|G|}$ partitions of the vertices into at most two parts. However for some structures we can prove the existence of *obstructions* with the property that every graph either has the structure or it has an obstruction, but not both. In this case, a lucky guess of an obstruction provides a proof that the structure does not exist. Theorem 22 is an example of this phenomenon.

Definition 18. A path, cycle, trail, walk W is *even* (*odd*) if its length is even (odd).

Lemma 19. *Every odd closed walk* $W = v_1 \dots v_n v_1$ *contains an odd cycle.*

Proof. Argue by induction on the length of W. If W is a cycle we are done. Otherwise there exist positive integers $i < j$ such that $v_i = v_j$. Then $W' := v_1 \dots v_i (= v_j) v_{j+1} \dots v_n v_1$ and $W'' := v_i v_{i+1} \dots v_j (= v_i)$ are shorter closed walks, both contained in W, and one of them must be odd. By the induction hypothesis, the odd one contains an odd cycle, which is also contained in W. □

Theorem 20. *A graph* $G = (V, E)$ *is bipartite iff it has no odd cycles.*

Proof. *Necessity.* Suppose G is bipartite with bipartition $\{A, B\}$. It suffices to show that if $C \subseteq G$ is a cycle then it is even. Since G is bipartite, $E(C) \subseteq E \subseteq E(A, B)$. So each edge $e \in E(C)$ has exactly one end in A. Thus the length of C is the even number

$$\|C\| = \sum_{e \in E} \sum_{v \in A \cap V(C)} \iota(v, e) = \sum_{v \in A \cap V(C)} \sum_{e \in E} \iota(v, e) = \sum_{v \in A \cap V(C)} d_C(v) = 2|A \cap V(C)|.$$

Sufficiency. Suppose G contains no odd cycle. It suffices to show that each component of G is bipartite (why?). So consider a component H of G, and let $x \in V(H)$. Set

$$A := \{v \in V(H) : \text{ there exists an odd } x, v\text{-walk in } H\} \text{ and}$$

$$B := \{v \in V(H) : \text{ there exists an even } x, v\text{-walk in } H\}.$$

Since H is a component of G, it is connected, and so $A \cup B = V(H)$. If there exists $v \in A \cap B$ then there exists an odd x,v-walk P and an even x,v-walk Q. Then $W = xPvQ^{*}x$ is an odd closed walk. By Lemma 19 there exists an odd cycle $C \subseteq W \subseteq G$. Since this is impossible, $A \cap B = \emptyset$. Similarly, if $uv \in E \smallsetminus E(A,B)$ then there exist paths xPu and xQv with the same parity. Thus $W = xPuvQ^{*}x$ is an odd closed walk; so there exists an odd cycle $C \subseteq W \subseteq G$, another contradiction. We conclude that $\{A,B\}$ is a bipartition of G. □

A.1.8 Dirac's Theorem

Let $G = (V,E)$ be a graph, and suppose $A,B \subseteq V$. An *A,B-walk* is a walk whose first vertex is in A, whose last vertex is in B and whose interior vertices are in neither A nor B. If $A = \{a\}$ or $B = \{b\}$, we may shorten this notation to an a,B-walk or an A,b-walk. Similarly, if $A = B$, we may shorten it to a B-walk. Also, if $H \subseteq G$ an H-walk is a $V(H)$-walk, etc.

The *minimal degree* of G is $\delta(G) = \min\{d(v) : v \in V\}$. Similarly, the *maximum degree* of G is $\Delta(G) = \max\{d(v) : v \in V\}$.

An *embedding* of H into G is an isomorphism from H to a subgraph of G. If there exists an embedding of H in G then we say that H can be *embedded* in G, or that H is embeddable in G. A subgraph $H \subseteq G$ is said to be a *spanning* subgraph of G if $V(H) = V(G)$. A spanning cycle of G is called a *hamiltonian* cycle. If G contains a hamiltonian cycle, G is said to be *hamiltonian*.

Many questions in graph theory have the following form: Given two graphs G and H with $|H| = |G|$ what "local" conditions on G ensure that H is embeddable in G? If G is complete then trivially H is embeddable in G. This is guaranteed by the local condition $\delta(G) = |G| - 1$. But in many cases we can do better. Corollaries 22, 23, below are examples.

Theorem 21. *Every connected graph $G = (V, E)$ with $|G| \geq 3$ contains a path or cycle of length at least $l = \min\{|G|, d(x) + d(y) : xy \notin E\}$.*

Proof. Let $P = v_1 \dots v_t$ be a path in G with maximum length. Then $N(v_1), N(v_t) \subseteq V(P)$, since otherwise we could extend P to a longer path. If P has length $\|P\| \geq l$ then we are done. Otherwise we prove:

$$G \text{ contains a cycle } C = (u_1 \dots u_t) \text{with } V(C) = V(P). \tag{A.1.4}$$

If $v_1 v_t \in E$ then $C := v_1 P v_n v_1$ is the desired cycle. Otherwise, $v_1 v_t \notin E$ and so

$$t = \|P\| + 1 \leq l \leq d(v_1) + d(v_t). \tag{A.1.5}$$

Let

$$X = \{i \in [t] : v_1 v_{i+1} \in E\} \text{ and } Y = \{i \in [t] : v_t v_i \in E\}.$$

Then $|X| = d(v_1)$, $|Y| = d(v_t)$, and $X \cup Y \subseteq [t-1]$. By (A.1.5) and inclusion-exclusion:

$$t - 1 \geq |X \cup Y| = |X| + |Y| - |X \cap Y| \geq t - |X \cap Y|$$
$$|X \cap Y| \geq 1.$$

Let $i \in X \cap Y$. See Figure **??**. Then $C = v_1 v_{i+1} \dots v_t v_i \dots v_1$ is a cycle that spans P, proving (A.1.4).

Since $l \leq |G|$ and $|C| = \|C\|$, it suffices to show $|G| = |C|$. Otherwise, there exists $x \in V - V(C)$. Since G is connected, there is an x, C-path $Q = x \dots u_1$. Choose notation so that $C = u_1 \dots u_t u_1$. Then $P' = Q u_1 u_2 \dots u_t$ is a longer path than P, a contradiction. □

Corollary 22 (Dirac's Theorem (1952))**.** *If $\delta(G) \geq \frac{1}{2}|G| > 1$ then G is hamiltonian.*

Proof. By Theorem 21, it suffices to prove that G is connected. Consider any distinct vertices x and y. Then

$$|G| \geq |N[x] \cup N[y]| = |N[x]| + |N[y]| - |N[x] \cap N[y]| \geq |G| + 2 - |N[x] \cap N[y]|$$

$$|N[x] \cap N[y]| \geq 2.$$

So x is connected to y by a path of length at most 2. □

Here is a weaker, but slightly less "local" condition that also ensures a graph is hamiltonian.

Corollary 23 (Ore's Theorem (1960))**.** *If G is a graph with $d(x) + d(y) \geq |G| \geq 3$ for all distinct nonadjacent vertices x an y then G is hamiltonian.*

Proof. We must show that G is connected. If x is adjacent to y they are connected. Otherwise apply the above argument. □

A.1.9 Even graphs and Euler's Theorem

Definition 24. A graph is *Eulerian* if it has a closed trail containing all edges. (Note that $T = v$ is closed, since its only vertex is its first and its last.) Such a trail is said to be an Eulerian trail. An *even* graph is a graph whose vertices all have even degree.

For $H \subseteq G$ and $v \in V(G) \setminus V(H)$, set $d_H(v) := 0$.

Fact 25. *If H and G are even graphs with $H \subseteq G$ then $H' := G - E(H)$ is even.*

Proof. Since G and H are even, every $v \in V(G)$ satisfies

$$d_{H'}(v) = d_G(v) - d_H(v) \equiv 0 \mod 2.$$

□

Proposition 26. *Let $T = v_1 ... v_t$ be a trail in a graph G. Then $d(v) \equiv 0 \mod 2$ for every vertex v, except that if T is open then $d(v_1) \equiv 1 \equiv d(v_t) \mod 2$.*

Proof. If T is open then set $T' := Tv_nv_1$; otherwise $T' := T$. Regardless, T' is closed. It suffices to show that every vertex v_i satisfies $d_{T'}(v_i) \equiv 0 \mod 2$, since $d_T(v_i) \not\equiv d_{T'}(v_i) \mod 2$ if and only if $i \in \{1,n\}$ and $v_1 \neq v_n$. We argue by induction on t.

If T' is a cycle, or $t = 1$, then every vertex v_i satisfies $d_{T'}(v_i) \equiv 0 \mod 2$. Otherwise, there exist $1 < i < j \leq t$ with $v_i = v_j$. Let $v_1T_1 = T'v_i + v_jT'v_1$ and $T_2 = v_iT'v_j$. Then T_1 and T_2 are both closed and every edge of T' is in exactly one of T_1 and T_2. By the induction hypothesis applied to T_1 and T_2,

$$d_{T'}(v) = d_{T_1}(v) + d_{T_2}(v) \equiv 0 + 0 \equiv 0 \mod 2.$$

□

Theorem 27 (Euler (1736))**.** *A graph G is Eulerian iff it has at most one nontrivial component and it is even.*

Proof. Necessity. Suppose G has a Eulerian trail T. Since T is connected it only contains edges from one component. Since T contains all edges, G has only one nontrivial component. Since T is closed and contains all edges of G, Proposition 26 implies every vertex of G has even degree (possibly 0).

Sufficiency. Suppose that G has at most one nontrivial component H and every vertex has even degree. Let $T = v_1 \dots v_t$ be a maximum length trail in G. Then T is closed: Otherwise v_t is incident to an odd number of edges of T by Proposition 26. Since $d(v_t)$ is even it is incident to some edge v_tv that is not in T. So we can extend T to $T^+ = v_1Tv_tv$, contradicting the maximality of T.

It remains to show that $E(H) \subseteq E(T)$. Otherwise there is an edge $ab \in E(H) \smallsetminus E(T)$. Since H is connected there is an $\{a,b\},T$-path P with no edge in T. Choose notation so that $P = b \dots v_i$. By definition, $a \notin V(P)$ (but maybe $b = v_i$). Since T is closed, $T^+ = abPv_iTv_i$ is a longer trail than T, a contradiction. $\square$

Lemma 28. *Every graph G with $\delta(G) \geq 2$ contains a cycle.*

Proof. Let $P = v_1 \dots v_t$ be a maximum path in G. Then $N(v_t) \subseteq V(P)$. So there exist $i < t-1$ such that $v_tv_i \in E(G)$. Thus $v_iPv_tv_i$ is a cycle contained in G. $\square$

Corollary 29. *If G is an even graph with $\|G\| > 0$ then G contains a cycle.*

Proof. Some component H of G contains an edge. Since H is connected, $\delta(H) \geq 1$. Since G is even this can be strengthened to $\delta(H) \geq 2$. So by Lemma 28, $H \subseteq G$ contains a cycle. $\square$

Second proof of Theorem 27 (Sufficiency). Suppose G is even and has at most one nontrivial component G'. We argue by induction on $\|G'\|$. If G' is a cycle or $\|G'\| = 0$, then the cycle or any vertex is the Eulerian trail.

Otherwise, by Corollary 29, G' contains a cycle C. Let H be a nontrivial component of $G' - E(C)$ (maybe $H = G' - E(C)$), and set $H' = G' - E(H)$. Both H and H' are even. Also H' is connected, since all components of $G' - E(C)$ that are contained in H' are connected to each other in H' by edges of C. Moreover, $\|H\| = \|G'\| - \|C\| < \|G'\|$ and $\|H'\| = \|G'\| - \|H'\| < \|G'\|$. So H and H' are even connected graphs with fewer edges than G'. By the induction hypothesis H and H' contain trails Eulerian trails T and T'. Moreover, T contains a vertex $v_1 \in C$ and T' contains all vertices of C. Choose notation so that $T = v_1 \dots v_n v_1$ and $T' = v_1 u_2 \dots u_m v_1$. Then $v_1 T v_n v_1 T' u_m v_1$ is an Eulerian trail in G', and G. $\square$

Theorem 30. *A connected graph G with exactly q vertices of odd degree decomposes into* $\max\{1, \frac{q}{2}\}$ *trails.*

Proof. By Lemma 7, q is even. Let G^+ be the result of adding a new vertex v^+ to G so that $N(v^+)$ is the set of vertices with odd degree in G. Since q is even, and every $v \in V(G)$ satisfies $d_{G^+}(v) \equiv d_G(v) + 1 \mod 2$ if and only if $d_G(v)$ is odd, G^+ is even. By Theorem 27, G^+ has an Eulerian Trail T. Removing v^+ partitions T into $\frac{q}{2}$ trails that decompose G. □

Alternatively, we could have proved Theorem 30 by adding q edges connecting disjoint pairs of odd degree vertices.

A.2 Cut-vertices, -edges and trees

Definition 31. A cut-vertex is a vertex in a graph G is a vertex such that $G - v$ has more components than G. Similarly, a cut-edge is an edge of G such that $G - e$ has more components than G.

Theorem 32. *An edge $e = xy$ in G is not a cut-edge iff it belongs to a cycle.*

Proof. Necessity. Suppose e is on a cycle C in a component H of G. It suffices to show that there exists a u,x-walk W in $H - e$ for all $u \in V(H)$. Since H is connected it contains a u,x-walk W'. If $e \notin W'$ then set $W := W'$; otherwise set $W := uWy(C-e)x$. Here we have used that $C - e$ is a path.

Sufficiency. Suppose e is not a cut edge. Then there exists an x,y-path P in $G - e$. Then $xPyx$ is a cycle in G. □

Theorem 33. *If $P = v_1 \dots v_t$ is a maximal path in G with $t \geq 2$ then its ends are not cut-vertices of G.*

Proof. By maximality, $N[v_1] \subseteq P$. Suppose u is a vertex in the component H of G containing v_1. It suffices to show that there exists a u,v_2-walk W in $H - v_1$. Since H is connected, there exists a u,v_2 walk W' in H. If $v_1 \notin W$ then set $W := W'$. Otherwise the predecessor of v_1 on Q is a vertex $v \in N(v_1) \subseteq P$; set $W := uW'vP^*v_2$. (Here P^* is P in reverse order.) □

Definition 34. A graph is *acyclic* if it contains no cycle. Acyclic graphs are also called forests. A connected acyclic graph is called a *tree*. A leaf is a vertex v with $d(v) = 1$. We say that a graph G satisfies (A) if it is acyclic, (C) if it is connected, and (E), if $|G| = \|G\| + 1$.

Lemma 35. *A graph G with $\|G\| \geq 1$ has at least two leaves if it satisfies (A) or both (C) and (E).*

Proof. First suppose that G is acyclic. Let $P = v_1 \dots v_t$ be a maximum path in G. Since G has an edge, $v_1 \neq v_t$. Since P is maximum and acyclic $N(v_1) = \{v_2\}$ and $N(v_t) = \{v_{t-1}\}$. So v_1 and v_t are distinct leaves.

Now suppose that G satisfies (C) and (E). Since G is connected and has an edge, $\delta(G) \geq 1$. Since G satisfies (E),

$$\sum_{v \in V(G)} d(v) = 2\|G\| = 2|G| - 2.$$

So at least two vertices of G have degree 1, i.e., are leaves. □

Lemma 36. *Suppose G is a graph with a leaf l and $G' = G - l$. Then for each of the conditions (A), (C) and (E), G satisfies it iff G' satisfies it.*

Proof. Suppose G is acyclic. Since removing a vertex cannot create a cycle G' is acyclic. Now suppose G' is acyclic. Since every vertex in a cycle has degree 2, adding a leaf l cannot create a cycle, and so G is acyclic.

Suppose G' is connected. Since l has a neighbor in $V(G')$, G is connected. Now suppose G is connected. Since $d(l) = 1$, there is a maximal path P with an end l. Thus l is not a cut-vertex, and so G' is connected.

Since $|G| = |G'| + 1$ and $\|G\| = \|G'\| + 1$, G satisfies (E) iff G' does. $\square$

Theorem 37. *If a graph G satisfies at least two of the conditions (A), (C), and (E) then it satisfies all three.*

Proof. Argue by induction on $|G|$.

Base Step: $|G| = 1$. By inspection, G satisfies all of (A), (C) and (E).

Induction Step: $|G| \geq 2$. Since G satisfies (C) or (E), it has an edge. Since G satisfies (A) or both (C) and (E), it has a leaf l by Lemma 35. Let $G' = G - l$. By Lemma 36, G' satisfies at least two of the conditions, since G does. By the induction hypothesis G' satisfies all three of the conditions. Thus G does also (Lemma 36). $\square$

Corollary 38. *G is a tree iff there is exactly one path between any two vertices.*

Proof. Sufficiency. G is connected, since there is a path between any two of its vertices. Suppose G has a cycle C, and let $ab \in E(C)$. Then ab and $a(C - ab)b$ are distinct a,b-paths, a contradiction. So G is also acyclic.

Necessity. Suppose G is a tree. Since it is connected, it has a path between any two of its vertices. We must show this path is unique. Otherwise there exist distinct vertices x,y and distinct x,y-paths P,Q. Choose x,y so that $\|P\| + \|Q\|$ is as small as possible. By minimality, P and Q have no common internal vertices: if $z \in V(P \cap Q)$ then, since $P \neq Q$, either $xPz \neq xQz$ or $zPy \neq zQy$; thus x,z or z,y is a better choice than x,y. Hence $xPyQ^*x$ is a cycle, in contradiction to G being acyclic. $\square$

Corollary 39. *Let G be a tree. Then*

1. Removing an edge disconnects G. So every edge is a cut-edge.

2. Adding an edge to G creates a unique cycle.

3. Every connected graph contains a spanning tree.

Proof. (1) Let $e \in E(T)$ and $G' = G - e$. Since

$$|G'| = |G| = \|G\| + 1 = \left\|G'\right\| + 2,$$

G' is not a tree. Since removing an edge cannot create a cycle, G' is not connected.

(2) Let e be a new edge joining two vertices of G, and set $G' := G + e$. Since

$$|G'| = |G| = \|G\| + 1 = \left\|G'\right\|,$$

G' is not a tree. Since adding an edge cannot disconnect G, G' contains a cycle. Now suppose G' contains two distinct cycles C, D and let $e' \in E(C) \smallsetminus E(D)$. Set $G^* := G' - e'$. Since e' is on the cycle C it is not a cut-edge. So G^* is connected. Also $|G^*| = \|G^*\| + 1$. So G^* is a tree. But this is a contradiction: G^* contains the cycle D, since e' is not on D.

(3) Let G be a connected graph. Let T be a connected spanning subgraph of G with as few edges as possible. It exists because G is a candidate. Then every edge of T is a cut-edge, and so no edge of T is on a cycle, i.e., T is acyclic. By the choice of T it is connected, spanning subgraph of G. So T is a spanning subtree of G.

(3, another proof) Let T be an acyclic spanning subgraph of G with as many edges as possible. It exists because the empty spanning subgraph is a candidate. By definition T is spanning and acyclic. It remains to show that it is connected. Since G is connected, it suffices to show that for any edge $uv \in E(G - T)$ there exists a u,v-path in T. By the maximality of T, $T + uv$ contains a cycle C. Since T is acyclic, $uv \in C$. Thus $u(C - uv)v$ is a uv-path in T.

(3, a third proof) Let T be a maximum subtree of G, i.e., a connected acyclic subgraph with as many vertices as possible. It suffices to show that T is spanning. Otherwise, there exists $v_1 \in V(G-T)$. Since G is connected there exists a v_1,T-path. Then $v_t \in V(T)$ and $v_{t-1} \notin V(t)$. So $T^+ = T + v_{t-1} + v_t v_{t-1}$ is a graph with a leaf v_{t-1}. By Lemma 36, T^+ is a tree. Since $T^+ \subseteq H$ and $|T^+| > |T|$, it contradicts the maximality of T.

(3, a fourth proof) Argue by induction on G. If $|G| = 1$ then G itself is a tree. So suppose $|G| > 1$. Let v be a non-cut-vertex. It exists by Lemma 33.Then $G' := G - v$ is connected. By the induction hypothesis it contains a spanning tree T'. Let $e \in E(v)$. It exists because G is connected and $|G| \geq 2$. Then $T := T' + v + e$ is a connected, spanning subgraph of G that satisfies $|T| = \|T\|$, and so T is a spanning tree of G. $\square$

Proposition 40. *Suppose T and T' are spanning trees of a graph G. Then for every $e \in E(T) \setminus E(T')$ there exists $e' \in E(T') \setminus E(T)$ such that $T - e + e'$ is a spanning tree of G.*

Proof. Let $e = ab$. Since T' is connected, it contains an a,b-path $P = (a =)v_1 \dots v_t(= b)$. Let i be the least index such that there is no a, v_i path in $T - ab$. Since ab is the unique ab path in T, t is a candidate for i, and so i exists. Clearly $i \neq 1$. Set $e' = v_{i-1}v_i$. Then $e' \notin E(T)$, and it is a cut-edge of $T^* := T - e + e'$. Since T is acyclic, and e', being a cut-edge, is not contained in a cycle of T^*, it follows that T^* is acyclic. By Theorem 37, T^* is a tree, since $|T^*| = |T| = \|T\| + 1 = \|T^*\| + 1$. $\square$

Proposition 41. *Suppose T and T' are spanning trees of a graph G. Then for every $e \in E(T) \setminus E(T')$ there exists $e' \in E(T') \setminus E(T)$ such that $T' + e - e'$ is a spanning tree of G.*

Proof. By Corollary 39(2), $T' + e$ contains a unique cycle C. Since T is acyclic, $C \nsubseteq T$, and so there exists an edge $e' \in E(C - T)$. Then $T^* := T' + e - e'$ is acyclic, since e' is an edge of the unique cycle in $T' + e$. By Theorem 37, T^* is a tree, since $|T^*| = |T| = \|T\| + 1 = \|T^*\| + 1$. □

Proposition 42. *If T is a tree with k edges and G is a nontrivial graph with $\delta(G) \geq k$ then G contains a copy of T, i.e., a subgraph isomorphic to T.*

Proof. Argue by induction on k.

Base Step: $k = 0$. Then $T \cong K_1$ so $T \cong G[\{v\}]$ for any vertex v.

Induction Step: $k > 1$. Let l be a leaf of T. Then $T' := T - l$ is a tree with $\|T'\| = k - 1$. By the induction hypothesis there exists $H' \subseteq G$ with $H' \cong T'$. Let p be the unique neighbor of l in T, and let x be the image of p in H'. Since $|H'| = \|H'\| + 1 = k$ and x is not adjacent to itself, x has at most $k - 1$ neighbors in H'. Since $\delta(G) \geq k$, there exists $y \in N_G(x) \smallsetminus V(H')$. Set $H = H' + y + xy$. Then $H \subseteq G$ and we can extend the isomorphism between T' and H' to an isomorphism between T and H by mapping l to y. □

A.3 Matchings

Definition 43. A *matching* is a set of edges with no common ends. A *maximal* matching is a matching that cannot be enlarged by adding an edge. A *maximum* matching is matching with maximum size among all matchings in the graph. A vertex is said to be *M-saturated* if and only if it is the end of an edge in M; otherwise it is *M-unsaturated*, and a set of vertices X is said to be *M-saturated* if every $x \in X$ is M-saturated. The matching M is *perfect* if every vertex is M-saturated

Definition 44. Given a matching M in a graph $G = (V, E)$, an *M-alternating* path is a path P such that each vertex $v \in V(P)$ is incident to at most one edge in $E(P) \smallsetminus M$. Such a path is *M-augmenting* if its ends are not M-saturated.

Theorem 45 (Berge)**.** *A matching M in a graph $G = (V, E)$ is not maximum in G iff G has an M-augmenting path.*

Proof. Suppose P is an M-augmenting path. Then

$$M' = M \triangle E(P) =_{def} (M \smallsetminus E(P)) \cup (E(P) \smallsetminus M)$$

is a larger matching.

Now suppose M is not maximum. Choose a maximum matching M'. Let H be the spanning subgraph with edge set $M \triangle M'$. Then $M \smallsetminus E(H) = M' \smallsetminus E(H)$. Since each vertex is incident to at most one edge of each matching, $\Delta(H) \leq 2$, and the components of H are alternating paths and even cycles. Since $|M| < |M'|$,

$$|M \cap E(H)| = |M| - |M \smallsetminus E(H)| < |M'| - |M' \smallsetminus E(H)| = |M' \cap E(H)|.$$

So one component of H has more edges from M' than M. Such a component must be an M-augmenting path. □

A.3.1 Bipartite matching

A bipartite G with bipartition $\{X, Y\}$ is called an X, Y-bigraph. For $S \subseteq X$ set $N(S) := \bigcup_{v \in S} N(v)$. For a function $f : A \to B$ and $S \subseteq A$, let $f(S) := \{y \in B : \exists x \in S(f(x) = y)\}$ be the range of f restricted to S.

Theorem 46 (Hall's Theorem [1935])**.** *An X, Y-bigraph G has a matching that saturates X iff*

$$|S| \leq |N(S)| \textit{ for all } S \subseteq X. \tag{A.3.1}$$

Proof. Suppose M is a matching, and let U be the set of M-unsaturated vertices in X. Set $m = \{(x,y) \in X \times Y : xy \in M\}$. Since the ends of M in X are distinct, m is a function with domain $X \smallsetminus U$. Since the ends of m in Y are distinct, m is an injection.

For any $S \subseteq X \smallsetminus U$, we have $m(S) \subseteq N(S)$, and since m is an injection, $|S| = |m(S)| \leq |N(S)|$. If M saturates X then $U = \emptyset$, and so A.3.1 holds.

Now suppose that no matching saturates X, and choose M to be maximum; so $U \neq \emptyset$. Let $A \subseteq V(G)$ be the set of ends of alternating paths starting in U. Set $S = A \cap X$ and $T = A \cap Y$. Then $U \subseteq S$ (witnessed by trivial paths). Consider any alternating path $P = v_0 \dots v_n$ with $v_0 \in U$. If i is even then $v_i \in S$, and if also $i \neq 0$ then $v_{i-1}v_i \in M$; if i is odd then $v_i \in T$. Since $|S \smallsetminus U| < |S|$, it suffices to show $|N(S)| \leq |T| \leq |S \smallsetminus U|$, or

$$\text{(i) } N(S) \subseteq T \text{ and (ii) } T \subseteq m(S \smallsetminus U). \tag{A.3.2}$$

Suppose $z \in N(S)$; say $wz \in E(S,z)$. Then there is an M-alternating path $Q = y_0 \dots y_{2k}$ with $y_0 \in U$, $w = y_{2k}$ and $y_{2k-1}w \in M$. Either $z \in V(Q)$ or Qwz is an alternating path starting in U. Regardless, $z \in T$. So $N(S) \subseteq T$, proving (A.3.2.i).

Suppose $z \in T$. Let $P = y_0 \dots y_{2k+1}$ be an M-alternating path with $y_0 \in U$, $z = y_{2k+1}$ and $y_{2k}z \notin M$. Since M is maximum, G has no M-augmenting path. So z is M-saturated; say $zx \in M$. Either $x \in V(P)$ or Pzx is an M-alternating path. Anyway, $x \in S$. Since $zx \in M$, we have $x \in S \smallsetminus U$; so $y \in m(S \smallsetminus U)$. Thus $T \subseteq m(S \smallsetminus U)$, proving (A.3.2.ii).

A graph is k-regular if every vertex has degree k. Recall that the degree of a vertex in a multigraph is the number (counting multiplicities) of edges incident to it. □

Corollary 47. *Every k-regular bipartite multigraph G has a perfect matching.*

Proof. Suppose G is an k-regular X,Y-bimultigraph. Then

$$k|X| = |E(X,Y)| = k|Y|.$$

It follows that $|X| = |Y|$. Thus it suffices to show that G has a matching that saturates X. By Hall's Theorem, it suffices to check A.3.1. Consider any subset $S \subseteq X$. Then

$$k|S| = |E(S,Y)| = |E(S,N(S))| \leq |E(X,N(S)| = k|N(S)|.$$

So $|S| \leq |N(S)|$. $\square$

Definition 48. A *vertex cover* of a graph G is a subset $Q \subseteq V(G)$ that contains at least one end of every edge.

Let C be an odd cycle with $\|C\| = 2k+1$. Since C is 2-regular, every m-set $Q \subseteq V(G)$ covers at most $2|Q|$ edges. Thus every vertex cover of C has at least $k+1$ vertices. On the other hand, every m-matching in C has $2m$ ends; so $m \leq k$.

Theorem 49 (Knig [1931], Egervry [1931])**.** *If $G = (V,E)$ is bipartite then the maximum size of a matching equals the minimum size of a vertex cover.*

Proof. Let G be a (not necessarily bipartite) graph, and choose an arbitrary orientation $\overrightarrow{G}$ of G. We first show that for *any* matching M *and cover* W $|M| \leq |W|$: Since W is a cover, every edge of M is incident to some vertex of W (possibly two). Define a function $g : M \to W$ by $g(e) \in e \cap W$, and if $e \subseteq W$ then $g(e)$ is the head of e in $\overrightarrow{G}$. Since M is a matching, no vertex of W can be incident two edges of M. So g is an injection. Thus $|M| \leq |W|$.

Now suppose G is an X,Y-bigraph and M is a maximum matching. Then it has no M-augmenting paths. Let $m, U, S, \overline{S}, T, \overline{T}$ be as in the proof of Hall's Theorem, and recall

$$\text{(i) } N(S) \subseteq T \text{ and (ii) } T \subseteq m(S \setminus U). \tag{A.3.3}$$

The set $W = \overline{S} \cup T$ is a vertex cover of G: Suppose $uv \in E = E(X,Y)$. If $v \in V \setminus W$ then $v \in S \cup \overline{T}$. By (A.3.3.1), if $v \in S$ then $u \in T$, and if $v \in \overline{T}$ then $u \in \overline{S}$.

It remains to show that $|W| \leq |M|$. Since $U \subseteq S$, every vertex in $\overline{S}$ is M-saturated. By (A.3.3.ii), every vertex in T is M-saturated. So (a) every vertex of W is M-saturated. Since m is an injection, (A.3.3.ii) implies $M \cap E(\overline{S}, T) = \emptyset$. So (b) no edge of M has both ends in W. Using (a) define a function $f : W \to M$ by $f(w) := e \in E(w) \cap M$ (e is unique, by the definition of matching). By (b) f is an injection. Thus $|W| \leq |M|$. □

A.3.2 General matching

Notice that if H is a component of a graph G and $|H|$ is odd then G does not have a perfect matching.

Definition 50. Let $\mathscr{C}_G$ be the set of components of the graph G. A component with an odd number of vertices is said to be an *odd component*. Let $\mathscr{O}_{\mathscr{G}}$ be the set of odd components of G and $o(G) = |\mathscr{O}_G|$. The graph G is factor critical if $G - v$ has a perfect matching for every vertex $v \in V(G)$. A set S is matchable into $\mathscr{O}_{G-S}$ if there exists a matching M such that each edge $e \in M$ has one end in S and one end in an odd component, and at most one vertex of each odd component is saturated.

Theorem 51 (Tutte [1947]). *Let $G = (V, E)$ be a graph with a maximum matching M. Then the number of M-unsaturated vertices of G is equal to*

$$\max_{S \subseteq V} o(G - S) - |S|.$$

Proof. For any set $S \subseteq V$ and matching M, there are at least $o(G - S) - |S|$ unsaturated vertices, since each odd component $H \subseteq G - S$ has an M-unsaturated vertex, unless $M \cap E(S, V(H)) \neq \emptyset$, and there are at most $|S|$ such edges in M. So it suffices to show that there exists a set $S \subseteq V$ and a matching M with exactly $o(G - S) - |S|$ unsaturated vertices.

Argue by induction on $|G|$. For the base step $|G| = 1$, let $S = \emptyset$. Then $o(G - S) - |S| = 1$ and the only vertex of G is unsaturated by any matching. Now consider the induction step.

Choose a set $S \subseteq V$ so that $o(G-S) - |S|$ is maximum, and subject to this, $|S|$ is also maximum. We will prove the following three claims:

Claim (1). Every component of $G-S$ is odd.

Proof. Suppose $H \in \mathscr{C}_{G-S}$ with $|H|$ even. Choose a non-cut vertex x (end of a maximal path) of H, and set $S' = S + x$. Then

$$\mathscr{O}_{G-S'} = \mathscr{O}_{G-S} + (H-x) \text{ and } |S'| = |S| + 1.$$

Thus $o(G-S) - |S| = o(G-S') - |S'|$, contradicting the choice of S, since $|S| < |S'|$. $\square$

Claim (2). Every odd component of $G-S$ is factor critical.

Proof. Consider any $H \in \mathscr{O}_{G-S}$ and any vertex $x \in V(H)$. We must show that $H' = H - x$ has a perfect matching. By the induction hypothesis, it suffices to show that $o(H'-T) - |T| \leq 0$ for all $T \subseteq V(H')$. So consider any such T, and set $S' = S \cup T + x$. Then $|S'| = |S| + |T| + 1 > |S|$, and so by the choice of S

$$o(G-S) - |S| > o(G-S') - |S'|.$$

Since $T + x \subseteq V(H)$,

$$\mathscr{O}_{G-S'} = (\mathscr{O}_{G-S} - H) \cup \mathscr{O}_{H'-T}.$$

Since $|S| < |S'|$, the choice of S implies $o(G-S) > o(G-S')$. Thus

$$o(G-S) - |S| > o(G-S') - |S'| = o(G-S) - 1 + o(H'-T) - |S| - |T| - 1$$

$$2 > o(H'-T) - |T|.$$

Moreover, by Claim (1), H is an odd component, and so $|H'|$ is even. Thus

$$0 \equiv |H'| \equiv o(H'-T) + |T| \mod 2.$$

Hence $1 \neq o(H'-T) - |T|$, and so $0 \geq o(H'-T) - |T|$. $\square$

Claim (3). S is matchable into $\mathscr{O}_{G-S}$.

Proof. Let H be the $S, \mathscr{O}_{G-S}$-bigraph with edge set

$$F := \{xD : x \in S, D \in \mathscr{O}_{G-S} \text{ and } N(x) \cap V(D) \neq \emptyset\}.$$

It suffices to show that H has a matching that saturates S. For this we apply Hall's Theorem.

Consider any set $T \subseteq S$. Since $|\emptyset| = 0 = |N(\emptyset)|$, we may assume that $T \neq \emptyset$. Thus $|S| > |S'|$, where $S' := S - T$. By the choice of S

$$o(G-S) - |S| \geq o(G-S') - |S'| \geq o(G-S) - N_H(T) - |S| + |T|$$

$$|N(T)| \geq |T|.$$

□

Finally, we obtain a matching M as follows. By Claim (3) there is a matching M_0 that saturates S and one vertex of $|S|$ odd components. For each $H \in \mathscr{O}_{G-S}$ choose a vertex v_H, and if possible, choose v_H so that it is M_0-saturated. Next use Claim (2) to obtain matchings M_H of $H - v_H$ for every odd component $H \in \mathscr{O}_{G-S}$. Then

$$M := M_0 \cup \bigcup_{H \in \mathscr{O}_{G-S}} M_H$$

is matching of G. Using Claim (1), it saturates every vertex of G except those $o(G-S) - |S|$ vertices v_H that are not saturated by M_0.

□

A.3.3 Applications of Matching Theorems

A graph is k-regular if every vertex has degree k. A *cubic* graph is a 3-regular graph. A k-regular spanning subgraph of a graph G is called a *k-factor* of G. Thus the edge set of a

1-factor of G is a perfect matching. A cut-edge is also called a *bridge*. A *bridgeless* graph is a graph without cut-edges. It need not be connected.

Theorem 52 (Petersen [1891]). *Every bridgeless cubic graph $G = (V,E)$ contains a* 1-*factor.*

Proof. By Tutte's Theorem, it suffices to show that $o(G-S) \leq |S|$ for every subset $S \subseteq V$. Fix any such S and consider any $H \in \mathscr{C}_{G-S}$. Since G is cubic and $|H|$ is odd,

$$3|H| = \sum_{v \in V(H)} d(v) = 2\|H\| + |E(V(H),S)| \equiv 1 \mod 2.$$

It follows that $|E(V(H),S)|$ is odd, and since G is bridgeless, $|E(V(H),S)| \geq 3$. Thus

$$3o(G-S) \leq |E(S,V \smallsetminus S)| \leq 3|S|,$$

and so $o(G \smallsetminus S) \leq |S|$. $\square$

Theorem 53 (Petersen [1891]). *Every regular graph with positive even degree has a* 2-*factor.*

Proof. Suppose $G = (V,E)$ is $2k$-regular with $k \in \mathbb{Z}^+$. It suffices to show that each component of G has a 2-factor, so we may assume G is connected. By Euler's Theorem 27, G has an Eulerian trail $T = v_1 \dots v_n (= v_1)$. Let D be an orientation of G obtained by directing each edge $e = xy$ as $\overrightarrow{e} = \overrightarrow{xy} := (x,y)$ iff there exists $i \in [n]$ such that $x = v_i$ and $y = v_{i+1}$.

Let $V' = \{v' : v \in V\}$ and $V'' = \{v'' : v \in V\}$ be sets of new vertices, disjoint from V and each other. Let H be the V',V''-bigraph defined by $x'y'' \in E(H)$ iff $\overrightarrow{xy} \in E(D)$. Then H is k-regular. So by the Corollary 47, H has a perfect matching M. Let $F = \{xy \in E : x'y'' \in M\}$. Then (V,F) is a 2-factor of G: for each $y \in V$ there exists a unique x such that $x'y'' \in M$ and a unique z such that $y'z'' \in M$. Since T is a trail, and $xy, yz \in T$, we have $xy \neq yz$. (If G were a multigraph then it could be that $x = z$, where xy and yz were parallel edges.) $\square$

A.4 Connectivity

Definition 54. A *separating set* or *vertex cut* of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. The *connectivity* $\kappa(G)$ is the minimum size of a vertex set S such that $G - S$ has more than one component or only one vertex. A graph G is k-connected if $k \leq \kappa(G)$.

Note that it is not possible to disconnect a complete graph by removing vertices. The definition sets the connectivity of a complete graph K equal to $|K| - 1$.

Definition 55. A *disconnecting set* of edges in a graph G is a set $F \subseteq E(G)$ such that $G - F$ has more than one component. The *edge-connectivity* $\kappa'(G)$ of G is the minimum size of a disconnecting set of edges. It is k-edge-connected if $k \leq \kappa'(G)$.

Following the text, we write $[S,T]$ for $E(S,T)$. An *edge cut* in G is a set of edges of the form $[S,\overline{S}]$, where $\emptyset \neq S \neq V(G)$ and $\overline{S}$ denotes $V(G) \setminus S$.

A.4.1 Basics

Theorem 56 (Whitney [1932])**.** *Every graph $G = (V,E)$ satisfies*

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

Proof. Choose a vertex v with $d(v) = \delta(G)$. Then $E(v)$ is a disconnecting set of edges of size $\delta(G)$, and so $\kappa'(G) \leq \delta(G)$.

For the first inequality, consider a minimum edge cut $[S,\overline{S}]$. Note that $\kappa(G) \leq |G| - 1$. First suppose that every vertex in S is adjacent to every vertex in $\overline{S}$. So

$$\kappa'(G) = |[S,\overline{S}]| = |S||\overline{S}| \geq |G| - 1 \geq \kappa(G).$$

Otherwise we can choose $x \in S$ and $y \in \overline{S}$ with $xy \notin E$. Set

$$T = (N(x) \cap \overline{S}) \cup \{w \in S - x : N(w) \cap \overline{S} \neq \emptyset\}.$$

Then every edge of $[S, \overline{S}]$ is incident to a vertex of T, and so every x,y-path contains a vertex of T. It follows that T is a separating set of G. Set $f : [S, \overline{S}| \to T$ by

$$f(wz) = \begin{cases} z & \text{if } w = x \\ w & \text{if } w \neq x \end{cases},$$

where $w \in S$. Clearly f is onto, and so

$$\kappa'(G) = |[S, \overline{S}]| \geq |T| \geq \kappa(G).$$

□

Theorem 57. *Every cubic graph $G = (V, E)$ satisfies $\kappa(G) = \kappa'(G)$.*

Proof. Put $\kappa := \kappa(G)$ and $\kappa' := \kappa'(G)$. If $\kappa = 0$ then G is disconnected, and so $\kappa' = 0$ also. If $\kappa = 3$ then $3 \leq \kappa \leq \kappa' \leq \delta(G) = 3$, and again $\kappa = \kappa'$. So assume $\kappa \in [2]$. Let S be a separating set with $|S| = \kappa$, and let H_1, H_2 be two components of $G - S$. Since S is minimum, every vertex $v \in S$ has a neighbor in each H_i. Since G is cubic, there exists i such that v has a unique neighbor w_v in H_i. Choose such a w_v, preferring $w_v \in H_1$, and set $F := \{vw_v : v \in S\}$. Then $|F| = |S|$. Moreover F is a disconnecting set of edges: F disconnects H_1 from S unless $w_v \notin H_1$ for some $v \in S$. In this case v has two neighbors in H_1, by the preference for H_1, and so no neighbors in S. Thus F separates $H_1 + v$ from H_2. So $\kappa' \leq |F| = |S| = \kappa \leq \kappa'$. □

Lemma 58 (Expansion Lemma)**.** *If G is k-connected and G' is obtained from G by adding a new vertex x with at least k neighbors in G then G' is k-connected.*

Proof. Since $|G'| = |G| + 1$, it suffices to show that G' does not have a $(k-1)$-separating set. Consider any $(k-1)$-set S. Then $G - S$ is connected and x has a neighbor in $G - S$, so $G' - S$ is connected. □

A.4.2 Low Connectivity

Definition 59. (This is not the same as the text.) Suppose H is a subgraph of G. Recall that a path $P \subseteq G$ is an H-path if its ends, but not it internal vertices are in H. Let $\mathscr{P} := P_0, P_1, \ldots, P_t$ be a sequence of subgraphs of G, and set $G_i := \bigcup_{j \in [i]} P_j$. Then $\mathscr{P}$ is a 2-*witness for* G if P_0 is a cycle, P_i is a G_{i-1}-path in G for all $i \in [t]$, and every $e \in E(G)$ is an edge of exactly one P_i.

Theorem 60 (Whitney [1932]). *A graph* $G = (V, E)$ *is* 2*-connected iff it has a* 2*-witness set.*

Proof. First suppose G is 2-connected. Then $\delta(G) \geq 2$, and so G contains a cycle C. Let $H \subseteq G$ be a maximal subgraph such that H has a 2-witness $P_0, \ldots, P_t$. It exists because C is a candidate. It suffices to show that $H = G$.

Suppose $v_0 \in V(G - H)$. Since G is connected, there exists a v_0, H-path $Q = v_0 \ldots v_s$. Since G is 2-connected, $G - v_s$ is connected. So there exists a v_{s-1}, H path P in $G - v_s$. Then $P_{t+1} := v_s v_{s-1} P$ is an H-path in $H + P_{t+1}$, contradicting the maximality of H. We conclude that H is a spanning subgraph of G.

Now suppose $xy \in E(G - H)$. Then xy is an H-path of $H + xy$, contradicting the maximality of H. So H is an induced, spanning subgraph of G. It follows that $G = H$.

Now suppose G has a 2-witness $P_0, \ldots, P_t$. Argue by induction on t that G is 2-connected. The base step $t = 0$ is easy since the cycle P_0 is 2-connected. So consider the induction step $t \geq 1$. By the induction hypothesis, $H := \bigcup_{i=0}^{t-1} P_i$ is 2-connected. Put $P_t := v_1 \ldots v_s$. Consider any $x \in V(G)$. We must show that $G - x$ is connected. Since H is 2-connected, $H - x$ is connected. Also every vertex of $P_t - x$ is connected to a vertex $v \in \{v_1, v_s\}$ in $G - x$, even if $x \in V(P_t)$. It follows that G is 2-connected. □

Definition 61. Let $e = xy$ be and edge in a graph G, and fix a new vertex v_e. The graph $G \cdot e$ obtained by contracting e is defined by

$$G \cdot e := G + K(v_e, N(\{x,y\})) - x - y.$$

Note that if P' is a path in $G \cdot e$ then either P' is a path in G or $v_e \in V(P')$. In the latter case we can obtain a path in G by replacing v_e by one of x, y, xy, yx. If P is a path in G then either P is a path in $G \cdot e$ or one or both of x, y are in $V(P)$. In the latter case we can obtain a path P' in $G \cdot e$ by replacing one of x, y, xPy, yPx by v_e.

Lemma 62 (Thomassen [1980])**.** *Every* 3*-connected graph* G *with* $|G| \geq 5$ *has an edge* e *such that* $G \cdot e$ *is* 3*-connected.*

Proof. Suppose not. Consider any edge xy. Since $G' := G \cdot xy$ is not 3-connected and $|G| \geq 4$, G' has a separating 2-set S. Observe that $v_{xy} \in S$: Otherwise v_{xy} is in a component H of $G' - S$ and $S \subseteq V(G)$. Choose w in another component of $G' - S$. We shall obtain a contradiction by showing that every $w, \{x,y\}$-path $P = w \dots u'u$ in G (with $u \in \{x,y\}$) contains a vertex of S, implying $\kappa(G) \leq |S| = 2$. Fix P; then $P' := wPu'v_{xy}$ is a w, v_{xy} path in G'. Since S separates w from v_{xy} in G', there exists $v \in V(P') \cap S \subseteq V(P)$.

So S has the form $S = \{v_{xy}, z\}$. Observe that $\{x,y,z\}$ is a separating 3-set in G: Choose vertices u, v in distinct components of $G' - S$. Then $u, v \in V(G)$. Consider any u, v-path $P \subseteq G$. If P contains neither x nor y then $P \subseteq G' - v_{xy}$, and so $z \in V(P)$.

So far $xy \in E$ and the 2 -separating set S are generic. Now choose xy and $S = \{v_{xy}, z\}$ as above so that $G - \{x,y,z\}$ has a component H that is as large as possible among all possible choices of xy and S. Let H' be another component of $G - \{x,y,z\}$. Since $\{x,y,z\}$ is a minimal separating set, each of x, y, z has a neighbor in each of H, H'. Let u be a neighbor of z in H'. Then $G \cdot uz$ has a separating set $\{v_{uz}, v\}$, and $\{u, v, z\}$ is a separating set for G.

Put $H^+ := G[V(H)+x+y]$. Then H^+ is connected, and $u, z \notin V(H^+)$. Thus $H^* := H^+ - v$ is disconnected, since otherwise H^* is a component of $G - \{u, v, z\}$ with $|H^*| = |H| + |\{x, y\} - v| \geq |H| + 1$, contradicting the choice of xy, S, H. Let U be a component of H^* containing neither x nor y (there are at least two components, and since x and y are adjacent, they are not in different components), and consider $a \in V(U)$. Since $a \in V(H)$, it has no neighbors in $G - \{x, y, z\} - H = G - H^+ - z$. Since $a \in U$, it has no neighbors in $H^+ - v - U$. Thus $N(a) \smallsetminus U \subseteq \{v, z\}$. So $\{v, z\}$ separates U from H', contradicting $\kappa(G) \geq 3$. $\square$

Definition 63. A sequence of graphs $G_0, \ldots, G_s$ is a 3-*witness for G* iff

1. $G_0 = K_4$ and $G_s = G$; and
2. for each $i \in [s]$ there is an edge $xy \in E(G_i)$ such that $G_{i-1} = G \cdot xy$ and $d_{G_i}(x), d_{G_i}(y) \geq 3$.

Theorem 64. *A graph G is* 3*-connected iff it has a* 3*-witness.*

Proof. First suppose that G is 3-connected. Then $|G| \geq 4$. We show by induction on $|G|$ that G has a 3-witness. Suppose $|G| = 4$. If $xy \notin E(G)$ then $V(G) \smallsetminus \{x, y\}$ is a 2-set that separates x from y, a contradiction. So $G = K_4$, and $G_0 = K_4 = G$ is a 3-witness for G. Otherwise, $|G| \geq 5$. By Lemma 62, there exists an edge $xy \in E(G)$ such that $G \cdot xy$ is 3-connected. Since G is 3-connected, $d(x), d(y) \geq 3$. By induction, $G \cdot xy$ has a 3-witness $G_0, \ldots, G_s$. So $G_0, \ldots, G_s, G$ is a 3-witness for G.

Now suppose $G_0, \ldots, G_s$ is a 3-witness for G. We show by induction on s that G is 3-connected. If $s = 0$ then $K_4 = G_0 = G$ is 3-connected. Otherwise, for some edge $xy \in E(G)$, both $G_{s-1} = G \cdot xy$ and $d_G(x), d_G(y) \geq 3$. Suppose for a contradiction that S is a 2-separator in G. If $S = \{x, y\}$ then v_{xy} is a cut vertex of $G \cdot xy$, a contradiction. So there is a component

H of $G-S$ that contains one, say x, of x and y, and another component H'. Since $d_G(x) \geq 3$ and $N(x) \subseteq V(H) \cup S$, x has a neighbor v in H (maybe $v = y$). If $y \notin S$ then S separates v from H' in $G \cdot xy$; otherwise $S' := S - y + v_{xy}$ separates v from H' in $G \cdot xy$. Regardless, we have a contradiction. $\square$

The last paragraph of the above proof is subtle. If $d_G(x) < 3$ then we could have $S = N(x)$, and $V(H) = \{x\}$. Then $H - x$ is not a component of $G \cdot xy - S'$ because $H - x$ has no vertices.

Conjecture 65 (Lovasz)**.** *There exists a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that for all $k \in \mathbb{Z}^+$ and $f(k)$-connected graphs G and all vertices $x, y \in V(G)$, there exists a partition $\{V_1, V_2\}$ of $V(G)$ such that $G[V_1]$ is an x, y-path and $G[V_2]$ is k-connected.*

A.4.3 Menger's Theorem

Definition 66. Let A and B be subsets of vertices in a graph $G = (V, E)$. An *A, B-path* is a path whose first vertex is in A, whose last vertex is in B and whose internal vertices are in neither A nor B. Let $l(A, B)$ be the maximum size of a set of disjoint A, B-paths. An *A, B-separating set (or A, B-separator)* is a set of vertices S such that $G - S$ has no A, B-paths. Let $k(A, B)$ be the minimum cardinality of an A, B-separating set.

Theorem 67 (Menger 1927)**.** *Let $G = (V, E)$ be a graph, and suppose $A, B \subseteq V$. Then the size $l := l(A, B)$ of a maximum set of disjoint A, B-paths is equal to the size $k := k(A, B)$ of a minimal A, B-separating set.*

Proof. Notice that $l \leq k$: If $\mathscr{P}$ *is a set of disjoint* A, B-paths and S is an A, B-separator then S must contain at least one vertex of each path, and no vertex is on more than one of the paths. Thus

$$
\begin{aligned}
f :& \mathscr{P} \to S \\
& P \mapsto \text{first } x \in S \cap V(P)
\end{aligned}
$$

is an injection, and so $|\mathscr{P}| \leq |S|$. Choosing $\mathscr{P}$ maximum and S minimum yields the inequality.

So it suffices to show $k \leq l$. Argue by induction on $\|G\|$.

Base Step: $\|G\| = 0$. Then every A,B-path is trivial. So $A \cap B$ is the maximum set of disjoint A,B-paths and the minimum A,B-separating set. Thus $l = |A \cap B| = k$.

Induction Step: $\|G\| \geq 1$. Let $e = xy \in E(G)$, and put $G' = G \cdot e$. For any $U \subseteq V$, define

$$U' = \begin{cases} U - \{x,y\} + v_e & \text{if } U \cap \{x,y\} \neq \emptyset \\ U & \text{otherwise} \end{cases},$$

and note that for every $T \subseteq V(G')$ there exists $S \subseteq V$ with $T = S'$. Every A',B' path P' in G' is an A,B-path in G, unless $v_e \in V(P')$, and in this case we can obtain an A,B path in G by replacing v_e by one of x, y, xy, yx. It follows that every set $\mathscr{P}'$ of disjoint A',B'-paths corresponds to a set $\mathscr{P}$ of disjoint A,B-paths with $|\mathscr{P}| = |\mathscr{P}'|$ (but not vice versa). So

$$l_{G'}(A',B') \leq l.$$

Also, if S is an A,B-separator in G if and only if S' is an A',B'-separator in G'. So

$$k_{G'}(A',B') \leq k \leq k_{G'}(A',B') + 1.$$

Choose a minimum A',B'-separator T in G'. If $k_{G'}(A',B') = k$ then by the induction hypothesis applied to G' we have:

$$k = k_{G'}(A',B') \leq l_{G'}(A',B') \leq l,$$

and we are done. Otherwise, $k = k_{G'}(A', B') + 1$. In this case $v_{xy} \in T$, and $T = S'$, where $S := T - v_{xy} + x + y$. In particular $xy \in G[S]$.

Set $G'' = G - e$. Since $e \in G[S]$,

$$k_G(A,S) = k_{G''}(A,S) \text{ and } k_G(B,S) = k_{G''}(B,S) \tag{A.4.1}$$

Since S separates A from B in G, every A,S-separator in G separates A from B, and so has size at least $|S|$, and a similar statement holds for B. So we have

$$k_G(A,S), k_G(S,B) \geq k. \tag{A.4.2}$$

Thus

$$|S| \geq l_G(A,S) \geq l_{g''}(A,S) =_{i.h.} k_{G''}(A,S) =_{(\text{A.4.1})} k_G(A,S) \geq_{(\text{A.4.2})} k = |S| \text{ and}$$

$$|S| \geq l_G(B,S) \geq l_{g''}(B,S) =_{i.h.} k_{G''}(B,S) =_{(\text{A.4.1})} k_G(B,S) \geq_{(\text{A.4.2})} k = |S|.$$

Let $\mathscr{K}_A$ be a collection of $|S| = k$ disjoint A,S-paths and $\mathscr{K}_B$ be a collection of $|S|$ disjoint S,B-paths. Then for each $z \in S$ there is a unique A,z-path P_z and a unique z,B-path Q_z. If $v \in V(P_w) \cap V(Q_z)$ then $v \in S$, since otherwise PvQ is an A,B-walk in $G - S$, contradicting the fact that S is an A,B-separator. Thus $w = v = z$, and so $\{P_z z Q_z : z \in S\}$ is a collection of $|S| = k$ disjoint A,B-paths. ☐

Definition 68. Let a and b be distinct vertices in a graph G. An a,b-separator is a set $S \subseteq V(G) - \{a,b\}$ such that there are no a,b-paths in $G - S$. Two a,b-paths are internally disjoint if they have no internal vertices in common. Let $\lambda(a,b)$ be the maximum number of internally disjoint a,b-paths and $\kappa(a,b)$ be the minimum size of an a,b-separating set (if it exists).

Corollary 69. *If x and y are nonadjacent vertices of a graph G then $\lambda(a,b) = \kappa(a,b)$.*

Proof. Let $A = N(a)$ and $B = N(b)$. Then any A,B-path $P' \subseteq G' = G - \{a,b\}$ can be extended to an a,b-path $P = aP'b \subseteq G$ and any a,b-path $Q \subseteq G$ contains an A,B-path $Q' = Q - a - b \subseteq G'$. Thus S is an a,b-separator of G iff it is an A,B-separator of G' and the maximum number of internally disjoint a,b-paths in G is equal to the maximum number of disjoint A,B-paths in G'. Applying Menger's Theorem to G' we see that the minimum size of an A,B-separating set is equal to the size of a maximum set of disjoint A,B-paths. The corollary follows. ☐

Theorem 70. *Every graph $G = (V,E)$ satisfies $\kappa(G) = t := \min_{a,b \in V} \lambda(a,b)$.*

Proof. Choose a,b so that $t = \lambda(a,b)$. First we show that $\kappa(G) \geq t$. If G is complete then

$$\lambda(a,b) = 1 + (|G| - 2) = |G| - 1 = \kappa(G),$$

since ab is an a,b-path, and acb is also an a,b-path for all $c \in V - a - b$. Otherwise G has a separating $\kappa(G)$-set S. Let x,y be vertices in distinct components of $G - S$. Then

$$t \leq \lambda(x,y) \leq |S| = \kappa(G).$$

Now we show that $t \geq \kappa(G)$. If $ab \notin E$ then by Corollary 69, there exists an a,b-separating set S with $|S| = t$; so $\kappa(G) \leq t$. Otherwise, $ab \in E$; set $G' = G - ab$. Then $\lambda_G(a,b) = \lambda_{G'}(a,b) + 1$, since ab is an a,b-path.

We first show that $\kappa(G') + 1 \geq \kappa(G)$. Suppose $k := \kappa(G') < \kappa(G)$. Then G' has a separating k-set S, and S is not a separating set of G. It follows that $G - S$ has exactly two components X and Y with (say) $a \in V(X)$ and $b \in V(Y)$. If $|X| > 1$ then $S + a$ is a separating of G with size $k + 1 \geq \kappa(G)$. A similar argument holds, if $|Y| > 1$. Otherwise $|G| = k + 2$, and so $\kappa(G) \leq |G| - 1 = k + 1$.

By Corollary 69, we have

$$\kappa(G) \leq \kappa(G') + 1 \leq \kappa_{G'}(a,b) + 1 = \lambda_{G'}(a,b) + 1 = \lambda_G(a,b) = t.$$

□

Definition 71. The line graph $H = L(G)$ of a graph $G = (V,E)$ is defined by

$$V(H) = E \text{ and } E(H) = \{ee' : e \cap e' \neq \emptyset\}.$$

Definition 72. Let x and y be distinct vertices in a graph G. An x,y-edge-cut is a set of edges F such that there are no x,y-paths in $G-F$; let $\kappa'(x,y)$ be the size of a minimum x,y-edge-cut. Two x,y-paths are edge-disjoint if they have no common edges; let $\lambda'(x,y)$ be the maximum size of a set of edge-disjoint x,y-paths.

Theorem 73. *Let $G = (V,E)$ be a graph with distinct vertices $x,y \in V$. Then $\kappa'(x,y) = \lambda'(x,y)$.*

Proof. Set $G' = G + x' + xx' + y' + yy'$, where x' and y' are new vertices. Then $\kappa'_G(x,y) = \kappa'_{G'}(x,y)$ and $\lambda'_G(x,y) = \lambda'_{G'}(x,y)$. A set of edges disconnects x from y in G' iff the corresponding set of vertices separates xx' from yy' in $L(G')$. Moreover, edge disjoint x,y-paths in G' correspond to internally disjoint xx',yy'-paths in $L(G')$. Thus

$$\kappa'_G(x,y) = \kappa'_{G'}(x,y) = \kappa_{L(G')}(xx',yy') = \lambda_{L(G')}(xx',yy') = \lambda'_{G'}(x,y) = \lambda'_G(x,y).$$

□

Definition 74. Let $G = (V,E)$ be a graph with $v \in V$ and $U \subseteq V$. An x,U-fan is a set of $|U|$ x,U-paths such that any two of paths share only the vertex x.

Theorem 75. *A graph $G = (V,E)$ is k-connected if and only if $|G| \geq k+1$ and G has an x,U-fan with $|U| = k$ for all $x \in V$ and all k-sets $U \subseteq V - x$.*

Proof. (Sketch) If G is k-connected then there exist k disjoint $N(x),U$-paths in $G-z$. Adding x yields the desired fan.

Conversely, the hypothesis implies $\delta(G) \geq k$, and for all x and y, there exist k disjoint $N(x), N(y)$-paths. It follows that

$$\kappa(G) = \min_{x \neq y \in V} \lambda(x,y) \geq k.$$

□

Theorem 76. *Let $G = (V,E)$ be a k-connected graph with $k \geq 2$. Then for any k-set $S \subseteq V$ there is a cycle $C \subseteq G$ with $S \subseteq V(C)$.*

Proof. Let $C \subseteq G$ be a cycle containing as many vertices of S as possible. It exists because $\delta(G) \geq \kappa(G) \geq 2$. We claim that $S \subseteq V(C)$. Otherwise, let $v \in V \smallsetminus V(C)$. Then $|S \cap V(C)| < k$. Arguing by contradiction, it suffices to find a cycle containing S$\cap V(C) + v$.

Orient C as $\overrightarrow{C}$. Let $t = \min\{k, |C|\}$, and let $\mathscr{F}$ be a $v, V(C)$-fan with $|\mathscr{F}| = t$. Set $F = \bigcup \mathscr{F}$, and let $x_1, \ldots, x_t$ be a sequence of the leaves of F in cyclic order around $\overrightarrow{C}$, and set $P_i = x_i \overrightarrow{C} x_{i+1}$. Then there exists $i \in [k]$ such that P_i contains no internal vertices from S: If $t = |C|$ this is true for all $i \in [k]$; otherwise it follows by the pigeonhole principle, since $t = k > |S \cap V(C)|$. So $D = x_{i+1} \overrightarrow{C} x_i F v F x_{i+1}$ is a cycle containing $S \cap V(C) + v$. □

A.5 Graph coloring

Definition 77. Let $G = (V,E)$ be a graph and C be a set (of colors). A *proper C-coloring* of G is a function $f : V \to C$ such that for all vertices $x, y \in V$ if $xy \in E(G)$ then $f(x) \neq f(y)$. If k is a positive integer, we say that f is a proper k-coloring if it is a proper $[k]$-coloring. The chromatic number $\chi(G)$ is the least k such that G has a proper k-coloring. In this case G is said to be k-chromatic. If G has a k-coloring then it is said to be k-colorable. In this chapter we will assume that all colorings are proper unless otherwise stated. For $i \in C$, $f^{-1}(i)$ is called a color class.

Proposition 78. *Every graph G satisfies* $\omega(G), \frac{|G|}{\alpha(G)} \leq \chi(G) \leq \Delta(G)+1$.

A.5.1 Examples

Example 79 (Mycielski [1955])**.** For every positive integer k there exists a graph G_k with $\omega(G_k) \leq 2$ and $\chi(G_k) = k$.

Proof. We argue by induction on k. For $k \leq 2$ let $G_k = K_k$. Now suppose $k \geq 3$ and we have constructed $G_{k-1} = (V_{k-1}, E_{k-1})$ as required. We first construct $G_k = (V_v, E_k)$ as follows: Let $V'_{k-1} = \{v' : v \in V_{k-1}\}$ be a set of new vertices, x_k be a new vertex, and put

$$V_k = V_{k-1} \cup V'_{k-1} + x_k \text{ and,}$$

$$E_k = E_{k-1} \cup \{uv' : uv \in E_{k-1}\} \cup \{x_k v' : v' \in V'_{k-1}\}.$$

So $N(v') \cap V_{k-1} = N(v) \cap V_{k-1}$ for all $v \in V_{k-1}$.

Suppose $\omega(G_k) \geq 3$, and choose $Q = K_3 \subseteq G_k$. Then $k \geq 3$. Since $N(x_k) = V'_{k-1}$ is independent, and $\delta(Q) = 2$, both $|V'_{k-1} \cap Q| \leq 1$ and $x_k \notin Q$. Since $\omega(G_{k-1}) =_{i.h.} 2$, there is exactly one $v' \in V'_{k-1} \cap Q$. Hence $N(v') \cap Q = N(v) \cap Q$, and so $Q - v' + v$ is a K_3 in G_{k-1}, a contradiction.

Notice that $\chi(G_k) \leq k$: If $k \leq 2$ this is obvious; otherwise G_{k-1} has a $(k-1)$-coloring f' by the induction hypothesis. We can extend f' to a k-coloring f of G_k by setting $f(v') = k$ (k is the new color), and $f(x_{k+1}) = 1$ (1 is an old color).

Finally we show that $\chi(G_k) \geq k$. If $k \leq 2$ this is obvious. For $k > 2$ it suffices to show that every $(k-1)$-coloring g of $G_k - x_k$ satisfies $g(V'_{k-1}) = [k-1]$, since then x_k will require a new color. Suppose not; say $\alpha \in [k-1] \smallsetminus g(V'_{k-1})$, and assume (wolog) that $\alpha = k-1$. Define a $(k-2)$-coloring h of G_{k-1} by:

$$h(v) = \begin{cases} g(v) & \text{if } g(v) \neq k-1 \\ g(v') & \text{if } g(v) = k-1 \end{cases}.$$

We claim that h is proper: Suppose $uv \in E_{k-1}$. Then $g(u) \neq g(v)$ since g is proper. If $g(u) \neq k-1 \neq g(v)$ then

$$h(u) = g(u) \neq g(v) = h(v).$$

Otherwise, at most one of u, v is colored with $k-1$ by g; say $g(v) = k-1$. Since $uv' \in E_{k+1}$,

$$h(u) = g(u) \neq g(v') = h(v).$$

This contradicts the induction hypothesis that $\chi(G_{k-1}) \geq k-1$. □

A.5.2 Brooks' Theorem

Lemma 80. *Let $G = (V,E)$ be a connected graph, and $v \in V$. There exists an ordering $v_1, \ldots, v_{|G|}$ of V such that (*) $v = v_{|G|}$ and for all $i \in [n-1]$ there exists $j \in [n] \smallsetminus [i]$ with $v_i v_j \in E$.*

Proof. Argue by induction on $n := |G|$. If $n = 1$ the only possible order works; so assume $n > 1$. Let P be a maximum path, and choose an end u of P with $u \neq v$. Since u is not a cut-vertex, $G - u$ is connected. By induction, there exists an ordering $v_2, \ldots, v_n$ of $V - u$ satisfying (*) for $G - u$. So $v_1 := u, v_2, \ldots, v_n$ satisfies (*) for G. □

Define a *b-obstruction* to be K_b, or, if $b = 3$, an odd cycle, and let (non-standard) $\omega^*(G)$ be the largest integer b such that G contains a b-obstruction. Then $\omega^*(G) \leq \chi(G) \leq \Delta(G) + 1$.

Theorem 81 (Brooks (1941))**.** *Every graph satisfies $\chi(G) \leq \max\{\omega^*(G), \Delta(G)\}$.*

Proof. Set $\Delta := \Delta(G)$, $\chi := \chi(G)$, $\omega^* := \omega^*(G)$, and argue by induction on $|G|$. Since $\chi \leq \Delta + 1$, it suffices to show that $\omega^* = \Delta + 1$ or $\chi \leq \Delta$. If $\Delta \leq 1$ then $\omega^* = \Delta + 1$. If $\Delta = 2$ then $2 \leq \omega^* \leq \chi \leq 3$; if $\omega^* = 2$ then $\chi \leq 2$ as G has no odd cycle. So assume $3 \leq \omega^* \leq \Delta$.

Let S be a maximal independent set, and put $G' := G - S$. So $\Delta(G') < \Delta$, since every vertex of G' has a neighbor in S. If $\omega^*(G') < \Delta$ then $(\Delta - 1)$-color G' by induction, and use a new color for S. Else, consider a Δ-obstruction $Q \subseteq G'$. Choose $y \in S$ with $\|y, Q\| \geq 1$. Then $V(Q) \not\subseteq N(y)$, since $G[Q+y] \neq K_{\Delta+1}$, and if $Q \neq K_\Delta$ then $\Delta < |Q|$. Since Q is connected, there is $wx \in E_Q(\overline{N}(y), N(y))$; let $w' \in N_Q(x) - w$.

Suppose there is a y, Q-path $P := y \ldots z$ in $G - x$. If $yw' \in E$, choose $P = yw'(= z)$; otherwise w' is a candidate for w. Regardless, we may assume $z \neq w$. Since $Q - w$ and $P - y$ are connected, $H := G[Q \cup P]$ has an ordering $L := w, y, v_1, \ldots, v_t, x$ such that each v_i has a neighbor to its right. Using induction, Δ-color $H' := G - H$ by f. Since

$$d_{H'}(w) + d_{H'}(y) \leq 2\Delta - d_H(w) - d_H(y) \leq 2\Delta - (\Delta - 1) - 2 < \Delta,$$

some color β is not used on $N(w) \cup N(y)$. Extend f to G by setting $f(w) = \beta = f(y)$ and coloring the remaining vertices in the order L. This is possible, since each v_i has at most $\Delta - 1$ colored neighbors when colored, and x has two neighbors w, y with the same color.

If there is no such P then $Q - x$ and y are in distinct components of $G - x$. Use induction to Δ-color $G - x$, so that y and $w \in Q \cap N(x)$ have the same color, and then color x. ☐

Conjecture 82 (Borodin & Kostochka 1977)**.** *If a graph G satisfies* $8, \omega(G) < \Delta(G)$ *then* $\chi(G) < \Delta(G)$.

A.5.3 Turn's Theorem

Definition 83. A graph is said to be *r-partite* if it is r-colorable. Saying r-partite instead of r-colorable tends to emphasize the partition into r independent sets provided by the r-coloring. These independent sets are called *parts.* The *complete r-partite $K_{n_1,\ldots,n_r}$* graph is the r-partite graph with r parts of sizes $n_1, \ldots, n_r$ such that any two vertices in different parts are adjacent. The *Turn* graph $T_{n,r}$ is the complete r-partite graph on n vertices such that any two parts differ in size by at most one.

Lemma 84. *Among all r-partite graphs on n vertices, $T_{n,r}$ has the most edges.*

Proof. Let G be an r-partite graph on n vertices with as many edges as possible. Suppose G has two parts X and Y with $|X|-|Y|\geq 2$, and $x\in X$. Let G' be the complete r-partite graph with the same parts as G except that X is replaced by $X-x$ and Y is replaced by $Y+x$. Then

$$E(G')\supseteq E(G)-\{xy:y\in Y\}+\{xx':x'\in X-x\}.$$

Thus

$$\|G'\|=\|G\|-|Y|+|X|-1\geq\|G\|+1,$$

a contradiction. So $G=T_{n,r}$. □

Theorem 85 (Turn [1941])**.** *Among all graphs $G=(V,E)$ on n vertices with $\omega(G)\leq r$, the one with the most edges is $T_{n,r}$.*

Proof. Evidently $T_{n,r}$ is a candidate. Argue by induction on r that if G satisfies $|G|=n$, $\omega(G)\leq r$, and $\|G\|\geq\|T_{n,r}\|$ then $G\cong T_{n,r}$. If $\omega(G)\leq 1$ then $G\cong T_{n,1}$; so suppose $r>1$.

Choose $v\in V$ with $d(v)=\Delta:=\Delta(G)$. Set $N:=N(v)$, $G':=G[N]$, $S:=V-N(v)$ and $G'':=G[S]$. Then $|G'|=\Delta$, and $\omega(G')\leq r-1$, since $K+v$ is a clique in G for every clique K in G'. Set $H:=T_{\Delta,r-1}\vee\overline{K}(S)$. Then H is an r-partite graph on n vertices, and $\omega(H)\leq r$, since any clique in H has at most $r-1$ vertices in $T_{\Delta,r-1}$ and one vertex in S. So

$$\begin{aligned}
\|G\| &= \|G'\| + \|G''\| + |E(N,S)| \\
&= \|G'\| + \sum_{v\in S} d_G(v) - \|G''\| \\
&\le \|T_{\Delta,r-1}\| + \sum_{v\in S} d_G(v) - \|G''\| && \text{(induction)} \quad \text{(A.5.1)} \\
&\le \|T_{\Delta,r-1}\| + \Delta|S| && \text{(maximum degree)} \quad \text{(A.5.2)} \\
&= \|H\| \\
&\le \|T_{n,r}\| && \text{(Lemma84)} \quad \text{(A.5.3)}
\end{aligned}$$

Inequality (A.5.1) is strict unless $G' \cong T_{\Delta,r-1}$. Inequality (A.5.2) is strict unless $G'' = \overline{K}(S)$ and $G = G' \vee G''$. Inequality (A.5.3) is strict unless $H \cong T_{n,r}$. If $\|G\| \ge \|T_{n,r}\|$ then all three inequalities are tight, and so

$$G \cong T_{\Delta,r-1} \vee \overline{K}(S) \cong H \cong T_{n,r}.$$

□

A.5.4 Edge Coloring

Definition 86. Let $G = (V,E)$ be a graph. A *proper k-edge-coloring* of G is a function $f : E \to [k]$ such that $f(e) = f(e')$ implies that e and e' are not adjacent ($e \cap e' = 0$). The *chromatic index* $\chi'(G)$ of G is the least k such that G has a proper k-edge-coloring. In this section we will assume that all edge colorings are proper. Note that this is not the case when we consider Ramsey Theory.

Theorem 87 (Knig [1916]). *Every bipartite graph G satisfies* $\chi'(G) = \Delta(G)$.

Proof. Argue by induction on $\Delta = \Delta(G)$. The base step $\Delta = 1$ is trivial since G has no adjacent edges, and so all edges can receive the same color. So consider the induction step

$\Delta > 1$. First observe that it suffices to find a Δ-regular bipartite multigraph H with $G \subseteq H$: By Hall's theorem H has a perfect matching M. Color all edges in $M \cap E(G)$ with color Δ, and set $G^* = G - M$. Then $\Delta(G^*) = \Delta - 1$, and so by the induction hypothesis, we can $(\Delta - 1)$-edge-color G^*. This yields a Δ-edge-coloring of G.

It remains to construct H. Suppose G has bipartition $\{A, B\}$ with $|A| \leq |B|$. Form G' by adding new vertices to A, but no new edges, to form a new set A' with $|A'| = |B|$. We will construct H so that it has bipartition $\{A', B\}$. Then we will have $\|H\| = \Delta|B|$. Argue by induction on $l(G') = \Delta|B| - \|G'\|$. If $l = 0$ we are done. Otherwise

$$\sum_{v \in A'} d_{G'}(v) = \|G'\| = \sum_{v \in B} d_{G'}(v) = \Delta|B| - l(G') > 0.$$

Thus there exist vertices $a \in A'$ and $b \in B$ such that $d_{G'}(a), d_{G'}(b) < \Delta$. Set $G^+ = G' + e$, where e is a new, possibly parallel edge, joining a and b. Then $l(G^+) < l(G')$, and so we are done by the induction hypothesis. $\square$

Now we consider edge coloring of general graphs. The fundamental result is Theorem 89 due to Vizing. The following lemma does most of the work in its proof.

Lemma 88. *Suppose $G = (V, E)$ is a simple graph with $\Delta(G) \leq k \in \mathbb{N}$, and $v \in V$. If $\chi'(G - v) \leq k$ and $d(x) = k$ for at most one $x \in N(v)$ then $\chi'(G) \leq k$.*

Proof. Argue by induction on k. If $k = 1$ then E is a matching, and so $\chi'(G) \leq 1$. Now suppose $k > 1$. For a function $f : E \to [k]$ and $\alpha \in [k]$, set

$$f(x) := [k] \smallsetminus \{f(e) : e \in E(x)\} \text{ and } f_\alpha := \{x \in N(v) : \alpha \in f(x)\}.$$

By adding edges and vertices to G, we may assume $k - 1 \leq d(x) \leq k = d(v)$ for all $x \in N(v)$, and $d(y) = k$ for exactly one $y \in N(v)$. So $|f(x)| = 2$ for all $x \in N(v) - y$ and $|f(y)| = 1$. Choose a k-edge-coloring f of $G' := G - v$ maximizing $T(f) := \{\beta \in [k] : 1 \leq |f_\beta| \leq 2\}$.

Suppose $|f_\alpha| \neq 1$ for all $\alpha \in [k]$. Since $\sum_{\alpha\in[k]} |f_\alpha| = \sum_{x\in N(v)} |f(x)| = 2k-1$, there exist $\beta, \gamma \in [k]$ with $|f_\beta| = 0$ and $|f_\gamma| \geq 3$; say $w \in f_\gamma$. Set $G_{\beta,\gamma} = (V - v, E_{\beta,\gamma})$, where $E_{\beta,\gamma} = \{e \in E : f(e) \in \{\beta, \gamma\}\}$. Then the component of $G_{\beta,\gamma}$ containing w is a path P with ends w and (say) z, where $f(z) \cap \{\beta, \gamma\} \neq \emptyset$. Obtain a new k-edge coloring f' of G' by exchanging colors γ and β on the edges of P. Then $f'(u) = f(u)$ for $u \in V(G') \smallsetminus \{w, z\}$, and $f'(w) = f(w) - \gamma + \beta$. Thus $w \in f'_\beta \subseteq \{w, z\}$. Hence $T(f) \subseteq T(f') + w$, a contradiction.

So $f_\alpha = \{z\}$ for some $z \in N(v)$ and $\alpha \in [k]$; say $\alpha = k$. Set $M = f^{-1}(k) + vz$. Since neither z nor v are incident to any edges colored k, M is a matching. Put $H := G - M$. Since $f_k = \{z\}$ and $vz \in M$, every vertex of $N[v]$ is M-saturated. So $d_H(x) \leq k-1$ for every $x \in N_H(v)$, and equality holds at most once. Since $f^{-1}(k) \subseteq M$, f is a $(k-1)$-coloring of $H - v$, and $\Delta(H - v) \leq k - 1$. So $\Delta(H) \leq k - 1$. By induction, $\chi'(G) \leq \chi'(H) + 1 \leq_{i.h.} k$. $\square$

Theorem 89 (Vizing (1964))**.** *Every graph $G = (V, E)$ satisfies $\chi'(G) \leq \Delta(G) + 1$.*

Proof. Set $k := \Delta(G) + 1$ and argue by induction on $|G|$. If $|G| = 1$ then $\chi'(G) \leq 1 = k$. Otherwise choose $v \in V$. By induction, $\chi'(G - v) \leq k$, and so by Lemma 88, $\chi'(G) \leq k$. $\square$

Theorem 90 (Full Vizing (1964))**.** *Every multigraph M satisfies $\chi'(M) \leq \Delta(M) + \mu(M)$.*

Conjecture 91 (Goldberg (1973), Seymour (1979))**.** *Every multigraph M with $\chi'(M) \geq \Delta(M) + 2$ satisfies $\chi'(M) \geq \max_{H\subseteq M} \lceil \frac{\|H\|}{\lfloor |V|/2 \rfloor} \rceil$.*

A.5.5 List Coloring

Definition 92. Let $G = (V, E)$ be a graph and C a set of colors. We write 2^C for the power set of C. A *list assignment* for G is a function $f : V \to 2^C$. One should think of $f(v) \subseteq C$ as the set of colors that are available for coloring the vertex v. A *k-list assignment* is a list assignment f such that $|f(v)| = k$ for all $v \in V$. Given a list assignment f, an f-coloring is a proper coloring g such that $g(v) \in f(v)$ for all $v \in V$. In this case G is f-colorable. The graph G is *k-list-colorable* (also *k-choosable*) if for every k-list assignment f it is f-colorable. The *list-chromatic number* (also *choosability*) $\chi_l(G)$ *of* G is the least k such that it is k-list colorable.

Example 93. Let $G = K_{t,t^t}$. Then $\chi(G) = 2$, but $\chi_l(G) \geq t+1$.

Proof. Let X, Y be a bipartition of G with $|X| = t$. Let f be a t-list assignment for G such that the vertices of X have disjoint lists of size t, and for each $\sigma \in \prod_{x \in X} f(x)$ there exists $y_\sigma \in Y$ with $f(y) = \text{range}(\sigma)$. Then for any f-coloring σ of $G[X]$, the vertex v_σ cannot be colored from the list $f(y_\sigma) = \sigma$. $\square$

Definition 94. An *edge-list assignment* for G is a function $f : E \to 2^C$. One should think of $f(e) \subseteq C$ as the set of colors that are available for coloring the edge e. A *k-edge-list assignment* is a list assignment f such that $|f(e)| = k$ for all $e \in E$. Given an edge-list assignment f, an f-coloring is a proper edge-coloring g such that $g(e) \in f(e)$ for all $e \in E$. In this case, G is f-list-colorable. The graph G is *k-edge-list-colorable* (also *k-edge-choosable*) if for every k-edge-list assignment f, it is f-colorable. The *list-chromatic index* $\chi'_l(G)$ *of* G is the least k such that it is k-edge-list colorable.

Conjecture 95. *Every graph* G *satisfies* $\chi'_l(G) = \chi'(G)$.

Definition 96. A *kernel* of a digraph $D = (V, A)$ is an independent set $S \subseteq V$ such that for every $x \in V \setminus S$ there exists $y \in S$ with $xy \in A$.

Lemma 97 (Bondy & Boppana & Siegel)**.** *Let* $D = (V, A)$ *be a digraph all of whose induced subgraphs have kernels. If* f *is a list assignment for* D *satisfying* $d^+(v) < |f(v)|$ *for all* $v \in V$ *then* D *has an* f*-coloring.*

Proof. Argue by induction on $|D|$. If $V = \emptyset$ the conclusion is vacuously true. Otherwise, fix $v_0 \in V$. Since $|f(v_0)| > d^+(v_0) \geq 0$, there exists $\alpha \in f(v_0)$. Set $W = \{v \in V : \alpha \in f(v)\}$. Then $v_0 \in W$. By hypothesis $D[W]$ has a (nonempty) kernel S. Color every vertex in S with α. This is possible because S is independent and $\alpha \in f(v)$ for every vertex in S.

Now it suffices to f-color $D' = D - S$ so that no vertex in $D - S$ is colored α. For this purpose, let f' be the list assignment for D' defined by $f'(v) = f(v) - \alpha$. Since $|D'| < |D|$, using induction, it suffices to show that $|f'(v)| > d^+_{D'}(v)$ for all $v \in V \smallsetminus S$.

If $v \notin W$ then $\alpha \notin f(v)$, and so

$$|f'(v)| = |f(v)| > d^+_D(v) \geq d^+_{D'}(v).$$

Else $v \in W$. Since S is a kernel of $D[W]$, there exists $w \in S = V \smallsetminus V(D')$ with $vw \in A$. So

$$|f'(v)| = |f(v) - \alpha| > d^+_D(v) - 1 \geq d^+_{D'}(v).$$

□

Theorem 98 (Galvin (1995))**.** *Every X,Y-bigraph G satisfies $\chi'_l(G) = \chi'(G) = \Delta(G)$.*

Proof. Let $\Delta := \Delta(G)$ and set $H := L(G)$. Then $\chi'_l(G) = \chi_l(H)$ and $\chi'(G) = \Delta$ (Theorem 87). Fix a Δ-edge coloring $c : E(G) \to [\Delta]$.

Let L be an Δ-edge-list assignment for G; so L is a Δ-list assignment for H. Our plan is to apply Lemma97 to H to show that it has an L-coloring f; then f is an L-edge-coloring of G. So it suffices to show H has an orientation $D := (E(G), A)$ such that (i) $\Delta^+(D) \leq \Delta - 1$ and (ii) every induced subgraph of D has a kernel.

Each $ee' \in E(H)$ satisfies $e \cap e' \subseteq X$ or $ee' \subseteq Y$, but not both. Define D by putting

$$ee' \in A \text{ iff } (e \cap e' \in X \wedge c(e) > c(e')) \vee (e \cap e' \in Y \wedge c(e) < c(e')).$$

Each $e \in E(G) = V(H)$ satisfies $d^+_H(e) \leq \Delta - 1$, since it has at most $c(e) - 1$ out-neighbors e' with $e \cap e' \in X$ and $\Delta - c(e)$ out-neighbors e' with $e \cap e' \in Y$. So (i) holds for D.

For (ii), consider any induced subgraph $D' := D[F]$, where $F \subseteq E(G) = V(D)$. Argue by induction on $|D'| \geq 1$. Let $X' := \{x \in X : E_G(x) \cap F \neq \emptyset\}$. For each $x \in X'$, choose $e_x \in E_G(x) \cap F$ with $c(e_x)$ minimum. Then $ee_x \in A$ for every $e \in E_G(x) \cap F - e_x$. If $Q = \{e_x : x \in X'\}$ is independent then it is a kernel of D'; else fix distinct $x, x' \in X'$ such that $e_x \cap e_{x'} \neq \emptyset$. So $e_x, e_{x'} \in E_G(y)$ for some $y \in Y$; say $c(e_x) < c(e_{x'})$. Let $D'' = D' - e_x$. By induction, D'' has a kernel S. If $e_{x'} \in S$ then S is a kernel for D', since $e_x e_{x'} \in A$. Otherwise, $e_{x'}e^* \in A$ for some $e^* \in S$. The choice of $e_{x'}$ implies $e_{x'} \cap e^* \notin X$. So $e_{x'} \cap e^* \in Y$, and thus $e, e', e^* \in E_G(y)$. So $c(e^*) > c(e_{x'})$, and also $c(e_{x'}) > c(e_x)$. Thus $e_x e^* \in A$, since $e_x \in E_G(y)$. Hence S is a kernel for D'. $\square$

A.6 Planar graphs

We have been informally drawing graphs in the Euclidean plane $\mathbb{R}^2$ since the start of the semester. Now we formalize the definition of a drawing of a graph in $\mathbb{R}^2$.

A.6.1 Very Basic Topology of the Euclidean Plane

Let $p, q \in \mathbb{R}^2$. The p,q-line segment $L_{p,q}$ is the subset of $\mathbb{R}^2$ defined by $L(p,q) := \{p + \lambda(q-p) : 0 \leq \lambda \leq 1\}$ and $\mathring{L}(p,q) := L(p,q) \smallsetminus \{p,q\}$. For distinct points $p_0, \dots, p_k \in \mathbb{R}^2$, the union $A(p_0, \dots, p_k) := \bigcup_{i \in [k]} L(p_{i-1}, p_i)$ is a (polygonal) p_0, p_k-arc provided $L(p_{i-1}, p_i) \cap \mathring{L}(p_{j-1}, p_j) = \emptyset$ for all distinct $i, j \in [k]$. We say that p_0 and p_k are linked by $A(p_0, \dots, p_k)$. If $A(p_0, \dots, p_k)$ is an arc and $L(p_0, \dots, p_k) \cap \mathring{L}(p_k, p_0) = \emptyset$ then $P(p_0, \dots, p_k, p_0) := L(p_0, \dots, p_k) \cup L(p_k, p_0)$ is a polygon. Note that arcs and polygons are closed in $\mathbb{R}^2$.

For $x \in \mathbb{R}^2$ the *open ball around x with radius r* is the set $B_r(x) := \{y \in \mathbb{R}^2 : \|x, y\| < r\}$. A set $U \subseteq \mathbb{R}^2$ is *open* if for all points $p \in U$ there exists $r > 0$ such that $B_r(x) \subseteq \mathbb{R}^2$. In particular, $\mathbb{R}^2$ and $\emptyset$ are open. The complement of an open set is a *closed* set. The *frontier* of a set X is the set of all points $y \in \mathbb{R}^2$ such that $B_r(x) \cap X \neq \emptyset$ and $B_r(x) \smallsetminus X \neq \emptyset$ for all $r > 0$. Note that if X is open, then its frontier lies in $\mathbb{R}^2 \smallsetminus X$.

Let U be an open set. Two points $x, y \in U$ are linked in U if there exists an x,y-arc contained in U. The relation of being linked is an equivalence relation on U. Its equivalence classes are called *regions*. Regions are open: Suppose $R \subseteq U$ is a region and $x \in R$. Then there exists a $r > 0$ such that $B_r(x) \subseteq U$. Clearly every $y \in B_r(x)$ is linked to x in U, since $L(x,y) \subseteq B_r(x)$. So $B_r(x) \subseteq R$. A closed set X separates a region R if $R \smallsetminus X$ has more than one region.

Suppose X is a finite union of points and arcs. Then $U := \mathbb{R}^2 \smallsetminus X$ is open. Let R be a region of U, and F be the frontier of R.

Theorem 99 (Jordan Curve Theorem for Polygons). *For every polygon $P \subseteq \mathbb{R}^2$, the set $\mathbb{R}^2 \smallsetminus P$ has exactly two regions. Each of these regions has the entire polygon as its frontier.*

A.6.2 Graph Drawings

Let $G = (V, E)$ be a graph. A *drawing* of G is a graph $\widetilde{G} := (\widetilde{V}, \widetilde{E})$ such that $\widetilde{V} \subseteq \mathbb{R}^2$, each edge $e \in \widetilde{E}$ is an arc between its ends, and $\widetilde{G} \cong G$. So edges are no longer just pairs of vertices, but have their own identity and structure (we need this anyway to formally deal with different edges linking the same two vertices). It should be clear that every finite graph has a drawing. Moreover, by moving vertices slightly and readjusting edges, we can (and do) require the following additional properties for drawings, without restricting the set of graphs that can be drawn.

1. No three edges have a common internal point.

2. The only vertices contained in an edge are its endpoints.

3. No two edges are tangent.

4. No two edges have more than one common internal point.

A *plane* graph is a drawing of a graph that has no crossing, i.e., two edges that have a common internal vertex. A *planar* graph is a graph that has a plane drawing.

Let $\widetilde{G} := (\widetilde{V}, \widetilde{E})$ be a plane graph. The faces of $\widetilde{G}$ are the regions of $\mathbb{R}^2 \setminus (\widetilde{V} \cup \bigcup \widetilde{E})$. The frontier of a face is called its boundary. Note that a plane cycle is a polygon. By the Jordan Curve Theorem we have:

Proposition 100. *A plane cycle is the boundary of exactly two faces.*

A.6.3 Basic facts

Fact 101 ()**.** *Let G be a planar multigraph. Removing a non-cut-edge reduces the number of faces by* 1*. Contracting a non-loop does not change the number of faces.*

Fact 102 ()**.** *Let G be a planar multigraph. Every non-cut-edge appears exactly once on the bounding walk of exactly two faces. Every cut-edge appears exactly twice on the bounding walk of exactly one face. In particular,*

$$\sum_{f \in F(G)} l(f) = 2\,\|G\|\,.$$

Theorem 103 (6.1.21 Euler's Formula (1758))**.** *Let G be a connected planar multigraph. Then*

$$|G| - \|G\| + |F(G)| = 2.$$

Proof. Argue by induction on $|G|$.

Base Step: $|G| = 1$. In this case all edges of G are loops. Argue by secondary induction on $\|G\|$. For the base step $\|G\| = 0$, note that G has one vertex and one face, and so

$$|G| - \|G\| + |F(G)| = 1 - 0 + 1 = 2.$$

For the induction step, set $G' = G - l$ for some loop l. Then using Fact 101:

$$|G| - \|G\| + |F(G)| = |G'| - (\left\|G'\right\| + 1) + (|F(G')| + 1) =_{i.h.} 2.$$

Induction Step: $|G| > 1$. Since G is connected, G has a non-loop edge e. Set $G' = G \cdot e$. Then

$$|G| - \|G\| + |F(G)| = (|G'| + 1) - (\left\|G'\right\| + 1) + |F(G')| = 2.$$

$\square$

Lemma 104 (). *Let $\widetilde{G}$ be a drawing of a* 2-connected *planar graph G. Then the boundary walk of every face is a cycle.*

Proof. By Theorem 60, G has an an ear decomposition $P_0, \ldots, P_h$. Argue by induction on h. For the base step $h = 1$, note that P_0 is a cycle that bounds the only two faces of $\widetilde{G}$ (Jordan Curve Theorem). Now consider the induction step. Let P_h have ends x, y, and note that $\mathring{P}_h$ is contained in some face f of $\widetilde{H} = \widetilde{G} - (P_h - x - y)$. By the induction hypothesis, every face of $\widetilde{H}$ is bounded by a cycle. Let $C = xv_1 \ldots v_a y v_{a+2} \ldots v_b x$ be the cycle that bounds f. Observe that $F(\widetilde{H}) - f \subseteq F(\widetilde{G})$. Using Euler's formula,

$$\begin{aligned}
|F(\widetilde{G})| &= 2 - |\widetilde{G}| + \left\|\widetilde{G}\right\| = 2 - (|\widetilde{H}| + |P_h| - 2) + (\left\|\widetilde{H}\right\| + \|P_h\|) \\
&= (2 - |\widetilde{H}| + \left\|\widetilde{H}\right\|) - (|P_h| - \|P_h\|) + 2 \\
&= |F(\widetilde{H})| - 1.
\end{aligned}$$

So $\mathring{P}_h$ divides $f \setminus \mathring{P}_h$ into exactly two new faces. By the Jordan Curve Theorem, one must be bounded by the cycle $xyv_{a+2} \ldots v_b x$ and the other by the cycle $xv_1 \ldots v_a yx$. $\square$

Theorem 105. *If G is a simple planar graph with at least three vertices then $\|G\| \le 3|G| - 6$. Moreover, if G has girth greater than* 3 *then $\|G\| \le 2|G| - 4$.*

Proof. We may assume that G is a maximal planar graph, i.e., it is not a spanning subgraph of any planar graph with more edges. Then G is connected, since otherwise we could add an edge between two components of G while maintaining planarity. Since G is connected and has at least three vertices, $\|G\| \geq 2$; since it is simple it has no parallel edges. Thus the length of every face is at least 3. By Fact 102,

$$2\,\|G\| = \sum_{f \in F(G)} l(f) \geq 3|F(G)|.$$

So $|F(G)| \leq \frac{2}{3}\,\|G\|$. By Theorem 103

$$2 = |G| - \|G\| + |F(G)| \geq |G| - \frac{1}{3}\,\|G\|\,,$$

and so $3|G| - 6 \geq \|G\|$.

Now suppose that the girth of G is greater than 3. Then every face boundary has length at least 4. So

$$2\,\|G\| = \sum_{f \in F(G)} l(f) \geq 4|F(G)|.$$

Thus $|F(G)| \leq \frac{1}{2}\,\|G\|$. By Theorem 103

$$2 = |G| - \|G\| + |F(G)| \geq |G| - \frac{1}{2}\,\|G\|\,,$$

and so $2|G| - 4 \geq \|G\|$. ☐

Corollary 106. *Neither K_5 nor $K_{3,3}$ is planar.*

Proof. If K_5 is planar then Theorem 105 yields the following contradiction:

$$10 = \|K_5\| \leq 3|K_5| - 6 = 9.$$

If $K_{3,3}$ is planar, then since it is bipartite, and so has girth greater than 3, Theorem 105 yields the contradiction:

$$9 = \left\|K_{3,3}\right\| \leq 2|K_{3,3}| - 4 = 8.$$

□

Theorem 107 (Kuratowski (1930))**.** *A graph is planar iff it contains neither a subdivision of K_5 nor a subdivision of $K_{3,3}$.*

We will break the proof of Kuratowski's Theorem into smaller pieces. First we need some preparation. Call a subdivision of K_5 or $K_{3,3}$ a *K-graph* (for Kuratowski).

Lemma 108 ()**.** *Let $e = xy$ be an edge of a graph G. If G contains no K-graph then $G \cdot e$ contains no K-graph.*

Proof. We prove the contrapositive. So suppose $G \cdot e$ contains a K-graph Q. Then $Q \subseteq G$ unless $v_e \in V(Q)$. So suppose $v_e \in V(Q)$. Note that $N_Q(v_e) \subseteq N_G(x) \cup N_G(y)$. First suppose that there exists $z \in \{x, y\}$ such that $|N_Q(v_e) \smallsetminus N_G(z)| \leq 1$. (This is the case if $d(v_e) \leq 3$.) Say $z = x$. Then replacing v_e by x or x, y, xy yields a subdivision of the K-graph Q. Otherwise, $d_Q(v_e) = 4$ and and $|N_Q(v_e) \smallsetminus N_G(z)| = 2$ for both $z \in \{x, y\}$. In this case Q is a subdivision of K_5. We show that G contains a subdivision of $K_{3,3}$! Let $Q' = Q - v_e + x + y + xy$, where x and y are new branch vertices, and (say) a, b, c, d are the remaining old branch vertices. Then Q' contains a subdivision $K_{3,3}$, where the branch vertices have the bipartition $\{\{x, a, b\}, \{y, c, d\}\}$. □

Corollary 109 ()**.** *If G is planar then G does not contain a K-graph.*

Proof. Suppose $Q \subseteq G$ is a K-graph. We show by induction on the number h of subdivision (degree 2) vertices in Q that G is not planar. The base step $h = 0$ is Corollary 106. For the induction step $h > 0$, consider a subdivision vertex x and one of its two neighbors y. Contracting xy yields a K-graph with one less subdivision vertex. Thus by the induction hypothesis $G \cdot xy$ is nonplanar, and so G is also nonplanar. □

Lemma 110 (). *Let $\widetilde{G}$ be a drawing of G with a face f. Then G can be redrawn as $\widetilde{H}$ so that the boundary of f is the boundary of the outer face in $\widetilde{G}$.*

Proof. Transfer $\widetilde{G}$ to a sphere, and then poke a hole in f. Stretching the resulting surface to the plane yields the desired drawing $\widetilde{H}$. $\square$

Let $G = (V, E)$ be a graph, and $S \subseteq V$. An S-lobe is a subgraph of the form $G[S \cup V(H)]$, where H is a component of $G - S$.

Lemma 111 (). *If G is nonplanar, but every proper subgraph of G is planar, then G is* 2*-connected.*

Proof. Certainly G is connected since we can draw an edge between the outer faces of any two components of G. Otherwise G has a cut-vertex v. Let H be a $\{v\}$-lobe of G and $H' = G - (H - v)$. Then by the minimality of G, H and H' are planar. Clearly we can attach a drawing $\widetilde{H}$ of H with v on the outer face boundary to a face of a drawing $\widetilde{H'}$ of H' whose boundary contains v. This violates the hypothesis that G is nonplanar. $\square$

Lemma 112 (Contrapositive, lobes). *Let G be a graph with subgraph $S = K_2 \subseteq G$. If G is nonplanar then there exists a nonplanar S-lobe of G.*

Proof. Argue by induction on the number h of components of $G - S$. The base step $h = 1$ is trivial; so consider the the induction step. Let H be an S-lobe, and $G' = G - (H - S)$. If H is nonplanar then we are done, and if G' is nonplanar we are done by induction. Otherwise let $\widetilde{H}$ be a drawing of H with S on the boundary of the outer face and $\widetilde{G'}$ be a drawing of G'. We can draw G by deforming $\tilde{H}$ to fit in a face of $\widetilde{G'}$ containing S, a contradiction. $\square$

Lemma 113. *If G is nonplanar and contains no K-graph (i.e., it is a counter example to Kuratowski's Theorem) and among such graphs $\|G\|$ is minimum then G is* 3*-connected.*

Proof. Suppose G is a minimum counterexample. Then G does not contain a K-graph and G is not planar. Thus no subgraph of G contains a K-subgraph. Since G is minimum, every proper subgraph of G is planar. So by Lemma 111, G is 2-connected.

Suppose for a contradiction G has a separating 2-set $S = \{x, y\}$; then it is a minimum separating set. Of course, $G^+ := G + xy$ is nonplanar. By Lemma 112, there exists a nonplanar S-lobe H^+ of G^+. Since S is a minimal separating 2-set, it has at least two neighbors in $G - H$. Thus $\|H^+\| \leq \|G^+\| - 2 < \|G\|$. So H^+ is not a counterexample. It follows that H^+ must contain a K-graph Q. All of Q appears in $H := G[V(H^+)]$ except xy. Let $H' \neq H$ be a component of $G - S$. Then H' contains an x,y-path P. So $Q - xy + P$ is a K-graph contained in G, a contradiction. $\square$

Theorem 114. *If G is a 3-connected graph that does not contain any K-graph then G is planar.*

Proof. We argue by induction on $|G|$. If $|G| \leq 4$ then G is planar, since $G \subseteq K_4$, and K_4 is planar. So suppose $|G| \geq 5$. By Theorem 62, G has an edge $e = xy$ such that $H = G \cdot e$ is 3-connected. By Corollary 108, H does not contain a K-graph. Thus by the induction hypothesis, H is planar. Let $\widetilde{H}$ be a drawing of H.

Let $H' = H - v_e$, and let f be the face of $\widetilde{H} - v_e$ that contains v_e. Since H' is 2-connected, Lemma 104 implies that f is bounded by a cycle C, say with orientation $\overrightarrow{C}$, and $(N_G(x) \cup N_G(y)) \smallsetminus \{x, y\} = N_H(v_e) \subseteq V(C)$. Choose the notation so that $d_G(x) \leq d_G(y)$. Obtain a drawing $\widetilde{G}'$ of $G' = G - y$ from $\widetilde{H}'$ by drawing x at the point corresponding to v_e in $\widetilde{H}$ and deleting the edges $v_e z$ with $z \in N_G(y) \smallsetminus N_G(x)$. Our goal is to extend $\widetilde{G}'$ to a drawing $\widetilde{G}$ of G by adding y and the edges in $E(y)$ to $\widetilde{G}'$.

Let $x_1, \ldots, x_k$ be the neighbors of x in G' arranged in cyclic order around $\overrightarrow{C}$, and $U := N_G(y) - x$. If there exists an index i such that $U \subseteq C_i := V(x_i \overrightarrow{C} x_{i+1})$ then we can extend $\widetilde{G}'$ to $\widetilde{G}$ by drawing y in the face f' of $\widetilde{G}'$ bounded by $xx_iCx_{i+1}x$, and then drawing edges from y to each vertex of $U+x$. This is possible, because all vertices of $U+x$ appear on the boundary of f'. Otherwise for every index i there exists $u_i \in U$ such that $u_i \notin C_i$.

If there exists an index i and a vertex $v_i \in N(y) \cap V(C_i - x_i - x_{i+1})$ then the vertices

$$v_i, u_i, x, x_i, x_{i+1}, y$$

are the branch vertices of a subdivision of $K_{3,3}$ with bipartition $\{\{v_i,u_i,x\},\{x_i,x_{i+1},y\}\}$, where the edges of the cycle $x_iv_ix_{i+1}u_ix_i$ are represented by the paths

$$x_i \overrightarrow{C} v_i, v_i \overrightarrow{C} x_{i+1}, x_{i+1} \overrightarrow{C} u_i, u_i \overrightarrow{C} x_i.$$

(Draw the picture.) This is a contradiction.

Otherwise, $U \subseteq N_G(x)$. Since $d_G(x) \leq d_G(y)$, we have $U = N_G(x) - y$. Thus $S := N_G(x) - y$ separates xy from the rest of G. Since $|G| \geq 5$ and G is 3-connected, $|S| \geq 3$. Say $a,b,c \in S$. Then x,y,a,b,c are the branch vertices of a subdivision of a K_5, where the edges of the cycle a,b,c,a are represented by the paths $a\overrightarrow{C}b, b\overrightarrow{C}c, c\overrightarrow{C}a$. (Draw the picture.) This is a contradiction. □

Proof of Kuratowski's Theorem 114. First suppose G contains a K-graph. Then by Lemma 109, G is not planar. Now suppose, for a contradiction, that some nonplanar graph does not contain a K-subgraph. Let G be such a graph with the minimum number of edges. By Lemma 113, G is 3-connected. But then by Theorem 114, G is planar, a contradiction. □

Theorem 115 (Thomassen (1994))**.** *Every simple planar graph G is 5-list colorable.*

Proof. It suffices to prove the following more technical statement by induction on G.

Claim. Suppose $\widetilde{G}$ is a drawing of a simple planar graph G such that every interior face, has length three, the boundary of the outer face is a cycle C, and $xy \in E$ is an edge of C. If L is a list assignment for G such that

1. $L(x) = \{\alpha\}$, $L(y) = \{\beta\}$, and $\alpha \neq \beta$,
2. $|L(v)| \geq 3$ for all vertices v on the outer boundary, except x and y, and
3. $|L(v)| \geq 5$ for all vertices not on the outer boundary,

then G has an L-coloring.

To see that the claim implies the theorem note that adding edges and vertices to G, and deleting colors from some lists of a list assignment L does not make it easier to L-color G. Moreover, we can add edges to a planar graph G so that every face is bounded by a C_3, and G remains planar. So it suffices to prove the claim.

Proof of Claim. Argue by induction on $|G|$. Note that $|G| \geq |C| \geq 3$. First consider the base step $|G| = 3$. Color x with α and y with β. The last vertex z has at least three colors in its list, and so we can color it with a color distinct from α and β.

Now consider the induction step $|G| > 3$. The outer face of $\widetilde{G}$ is bounded by a cycle $C = v_1 v_2 \dots v_s v_1$ with $v_1 = x$ and $v_2 = y$.

Case 1. C has a chord $v_i v_j$ with $i > j$. Let $C_1 = v_i v_{i+1} \dots v_j v_i$ and $C_2 = v_j v_{j+1} \dots v_i v_j$ be the two nonspanning cycles contained in $C + v_i v_j$. Let $\widetilde{G}_i$ be the plane graph formed by C_i and its interior. Then $\widetilde{G}_1 \cup \widetilde{G}_2 = G$, $\widetilde{G}_1 \cap \widetilde{G}_2 = \widetilde{G}[\{v_i v_j\}]$, and $x, y \in V(C_1)$. By the induction hypothesis, there exists an L-coloring f_1 of $\widetilde{G}_1$. Set $x' = v_i$, $\alpha' = f(x')$, $y' = v_j$, $\beta' = f(y')$, $L'(x') = \{\alpha'\}$, $L'(y') = \beta'$ and $L'(v) = L(v)$ for all vertices of $\widetilde{G} - x' - y'$. Then by the induction hypothesis there exists an L'-coloring g_2 of $\widetilde{G}_2$. It follows that $f = g_1 \cup g_2$ is an L-coloring of G.

Case 2. C does not have a chord. Since every interior face is bounded by a C_3, $G[N(v_s)]$ contains a hamiltonian x, v_{s-1}-path P. Moreover, since C has no chords, the outer face of

$\widetilde{G}' = \widetilde{G} - v_s$ is bounded by the cycle $C' = xPv_{s-1}C^*v_1(=x)$. Of course, the interior faces of $\widetilde{G}'$ have length three. Let $\gamma, \delta \in L(v_s)$ be distinct colors not equal to α. Define a list assignment L' for L by

$$L'(v) = \begin{cases} L(v) - \gamma - \delta & \text{if } v \in V(P) - x - v_{s-1} \\ L(v) & \text{else} \end{cases}.$$

By the induction hypothesis $\widetilde{G}'$ has an L' coloring f'. Choose $\varepsilon \in \{\gamma, \delta\}$ such that $\varepsilon \neq f'(v_{s-1})$. Then f' can be extended to an L-coloring f of G by setting $f(v_s) = \varepsilon$.

☐

This completes the proof of the claim and the Theorem. ☐

A.7 Extras

A.7.1 Lower Bounds on Ramsey's Theorem

Proof. Fix $k \geq 2$, and set $n = \lfloor 2^{k/2-1/2} \rfloor$. Let V be a set of n vertices, and $\mathscr{G}$ be the set of all graphs G with $V(G) = V$. So $G = (V,E) \in \mathscr{G}$ if and only iff $E \subseteq \binom{V}{2}$. Since there are $2^{\binom{n}{2}}$ choices for E,

$$|\mathscr{G}| = N := 2^{\binom{n}{2}}. \tag{A.7.1}$$

For $X \subseteq V$ with $|X| = k$, let $\mathscr{G}_X$ be the set of graphs in $\mathscr{G}$ such that X is a clique or co-clique. So if $G := (V,E) \in \mathscr{G}$ then $G \in \mathscr{G}_X$ iff $E \cap \binom{X}{2} \in \{\emptyset, \binom{X}{2}\}$ and $E \smallsetminus \binom{X}{2} \subseteq \binom{V}{2} \smallsetminus \binom{X}{2}$. There are two possibilities for the first conjunct and $2^{\binom{n}{2}-\binom{k}{2}}$ possibilities for the second. Thus

$$|\mathscr{G}_X| = 2 \cdot 2^{\binom{n}{2}-\binom{k}{2}} = 2N2^{-(k^2-k)/2}. \tag{A.7.2}$$

Any graph G in

$$\mathscr{G} \smallsetminus \bigcup_{X \in \binom{V}{k}} \mathscr{G}_X \neq \emptyset,$$

satisfies $\omega(G), \alpha(G) < k$ and $|G| = n$. So it suffices to prove $|\bigcup_{X \in \binom{V}{k}} \mathscr{G}_X| < |\mathscr{G}|$.

Since (a) $|\bigcup_{X \in \binom{V}{k}} \mathscr{G}_X| \leq \binom{n}{k} |\mathscr{G}_X|$, (b) $\binom{n}{k} < \frac{n^k}{k!}$, and (c) $\frac{n}{2^{k/2-1/2}} \leq 1$,

$$\begin{aligned} |\mathscr{G}| - |\bigcup_{X \in \binom{V}{k}} \mathscr{G}_X| &\geq N - \binom{n}{k} \cdot 2N2^{-(k^2-k)/2} && ((a),(\text{A.7.1}),(\text{A.7.2})) \\ &> N(1 - \frac{2}{k!} n^k 2^{-(k-1)k/2}) && (b) \\ &\geq N(1 - (\frac{n}{2^{k/2-1/2}})^k) \geq 0, && (c) \end{aligned}$$

and we are done. □

A.7.2 Equitable Coloring

Definition 116. An equitable k-coloring of a graph $G = (V, E)$ is a proper coloring $f : V \to [k]$ such that difference $||f^{-1}(i)| - |f^{-1}(j)||$ in the sizes of the the i-th and j-th color classes is at most 1 for all $i, j \in [k]$. In particular, every color is used if $|G| \leq k$.

Theorem 117 (Hajnal & Szemerédi Theorem (1976))**.** *Every graph G with maximum degree at most r has an equitable* $(r+1)$*-coloring.*

The proof was long and sophisticated, and does not provide a polynomial time algorithm. Kierstead and Kostochka found a much simpler and shorter proof. This better understanding has led to many new results, several of which are stated below.

Let $\theta(G) = \max\{d(x) = d(y) : xy \in E(G)\}$.

Theorem 118 (Kiestead & Kostochka (2008))**.** *For every* $r \geq 3$*, each graph G with* $\theta(G) \leq 2r+1$ *has an equitable* $(r+1)$*-coloring.*

Theorem 119 (Kierstead, Kostochka, Mydlarz & Szemerédi)**.** *There is an algorithm that constructs an equitable k-coloring of any graph G with* $\Delta(G) + 1 \leq k$*, using time* $O(r|G|^2)$*.*

Problem 120. Find a polynomial time algorithm for constructing the coloring in Theorem (118).

One might hope to prove an equitable version of Brooks' Theorem, but the following example shows that the statement would require special care: For r is odd, $K_{r,r}$ satisfies $\Delta(K_{r,r}) = r$ and $\omega(G) = 2$, but has no r-equitable coloring. Chen, Lih and Wu [**?**] proposed the following common strengthening of Theorem 117 and Brooks' Theorem.

Conjecture 121. *Let G be a connected graph with $\Delta(G) \leq r$. Then G has no equitable r-coloring if and only if either (a) $G = K_{r+1}$, or (b) $r = 2$ and G is an odd cycle, or (c) r is odd and $G = K_{r,r}$.*

Kierstead and Kostochka have proved the conjecture for $r \leq 4$, and also for $r \geq \frac{1}{4}|G|$.

Proof of Theorem 117. Let G be a graph with $\Delta(G) \leq r$. We may assume that $|G|$ is divisible by $r+1$: If $|G| = s(r+1) - p$, where $p \in [r]$ then set $G' := G + K_p$. Then $|G'|$ is divisible by $r+1$ and $\Delta(G') \leq r$. Moreover, the restriction of any equitable $(r+1)$-coloring of G' to G is an equitable $(r+1)$-coloring of G. So we may assume $|G| = (r+1)s$.

We argue by induction on $\|G\|$. The base step $\|G\| = 0$ is trivial, so consider the induction step. Let u be a non-isolated vertex. By the induction hypothesis, there exists an equitable $(r+1)$-coloring of $G - E(u)$. We are done unless some color class V contains an edge uv. Since $\Delta(G) \leq r$, some color class W contains no neighbors of u. Moving u to W yields an $(r+1)$-coloring of G with all classes of size s, except for one *small* class $V^- := V - u$ of size $s-1$ and one *large* class $V^+ := W + u$ of size $s+1$. Such a coloring is called *nearly equitable*.

Given a nearly equitable $(r+1)$-coloring, define an auxiliary digraph $\mathscr{H}$, whose vertices are the color classes, so that UW is a directed edge if and only if some vertex $y \in U$ has no neighbors in W. In this case we say that *y witnesses UW*. Let $\mathscr{A}$ be the set of classes from which V^- can be reached in $\mathscr{H}$, $\mathscr{B}$ be the set of classes not in $\mathscr{A}$ and $\mathscr{B}'$ be the set of classes reachable from V^+ in $\mathscr{H}[\mathscr{B}]$. Set $a := |\mathscr{A}|$, $b := |\mathscr{B}|$, $b' := |\mathscr{B}'|$, $A := \bigcup \mathscr{A}$, $B := \bigcup \mathscr{B}$ and $B' := \bigcup \mathscr{B}'$. Then $r+1 = a+b$. Since every vertex $y \in B$ has a neighbor in every class of $\mathscr{A}$ and every vertex $z \in B'$ also has a neighbor in every class of $\mathscr{B} - \mathscr{B}'$,

$$d_A(y) \geq a \text{ for all } y \in B \text{ and } d_{A \cup B \setminus B'}(z) \geq a+b-b' \text{ for all } z \in B'. \qquad (*)$$

Case 0: $V^+ \in \mathscr{A}$. Then there exists a V^+,V^--path $\mathscr{P} = V_1, \ldots, V_k$ in $\mathscr{H}$. Moving each witness y_j of V_jV_{j+1} to V_{j+1} yields an equitable $(r+1)$-coloring of G. $\square$

We now argue by a secondary induction on b, whose base step $b = 0$ holds by Case 0. Also $|A| = as - 1$ and $|B| = bs + 1$. Now consider the secondary induction step.

A class $W \in \mathscr{A}$ is *terminal*, if every $U \in \mathscr{A} - W$ can reach V^- in $\mathscr{H} - W$. Let $\mathscr{A}'$ be the set of terminal classes, $a' := |\mathscr{A}'|$ and $A' := \bigcup \mathscr{A}'$. An edge wz is *solo* if $w \in W \in \mathscr{A}'$, $z \in B$ and $N_W(z) = \{w\}$. Ends of solo edges are *solo* vertices and *solo neighbors* of each other.

Order $\mathscr{A}$ as $V^-, X_1, \ldots, X_{a-1}$ so that each X_i has a previous out-neighbor.

Case 1: For some $a - b \leq i \leq a-1$, class X_i is not terminal. Then some $X_j \in \mathscr{A}'$ cannot reach V^- in $\mathscr{H} - X_i$. So $j > i$ and X_j has no out-neighbors before X_i. In particular, $d^+_{\mathscr{A}}(X_j) < b$. Then for each $w \in X_j$, $d_A(w) \geq a - b$, and so $d_B(w) < 2b$. Let S be the set of solo vertices in X_j, and $D := X_j \setminus S$. If $v \in B - N_B(S)$ then v has no solo neighbor in X_j, and so has at least two neighbors in D. Thus $2b\,|D| > 2\,|B - N_B(S)|$. Using $|S| + |D| = s$ and $r\,|S| \geq |E(S,A)| + |N_B(S)|$,

$$bs + (a-1)\,|S| = b\,|D| + r\,|S| > |B - N_B(S)| + |E(S,A)| + |N_B(S)| > bs + |E(S,A)|\,.$$

Thus $(a-1)|S| > |E(S,A)|$, and so there exists $w \in S$ with $d_A(w) \le a-2$. Thus w witnesses some edge $X_jX \in E(\mathscr{H}[A])$. Since $w \in S$, it has a solo neighbor $y \in B$.

Move w to X and y to X_j. This yields nearly equitable colorings of $G[A+y]$ and $G[B-y]$. Since X_j is terminal, $X+w$ can still reach V^-. Thus by Case 0, $G[A+y]$ has an equitable a-coloring. By (*), $\Delta(G[B-y]) \le b-1$. So by the primary induction hypothesis $G[B-y]$ has an equitable b-coloring. After combining these equitable colorings we are done.

Case 2: All the last b classes $X_{a-b},\dots,X_{a-1}$ are terminal. Then $a' \ge b$. For $y \in B'$, let $\sigma(y)$ be the number of solo neighbors of y. Similarly to (*),

$$r \ge d(y) \ge a+b-b'+d_{B'}(y)+a'-\sigma(y) \ge r+1+d_{B'}(y)+a'-b'-\sigma(y).$$

So $\sigma(y) \ge a'-b'+d_{B'}(y)+1$. Let I be a maximal independent set with $V^+ \subseteq I \subseteq B'$. Then $\sum_{y\in I}(d_{B'}(y)+1) \ge |B'| = b's+1$. Since $a' \ge b$,

$$\sum_{y\in I}\sigma(y) \ge \sum_{y\in I}(a'-b'+d_{B'}(y)+1) \ge s(a'-b')+b's+1 > a's = \left|A'\right|.$$

So some vertex $w \in W \in \mathscr{A}'$ has two solo neighbors y_1 and y_2 in the independent set I.

Since the class Y of y_1 is reachable from V^+, we can equitably b-color $G[B-y_1]$. Let Y' be the new class of y_2. If w witnesses an edge WX of $G[A]$ then we are done by Case 1; otherwise we can move w to some class $U \subseteq B'-y_1$. Replacing w with y_1 in W to get W^* and moving w to U yields a new nearly equitable $(r+1)$-coloring of G. If $U \in \mathscr{A}$ then we are done by Case 0; otherwise at least $a+1$ classes, W^*, Z', and all $X \in \mathscr{A}'-W$, can reach V^-. In this case we are done by the secondary induction hypothesis.

A.8 Exceptional notation

$|G| = |V(G)| = n(G)$, number

$\|G\| = |E(G)| = e(G)$, number

$E(v) = \{vu : vu \in E\}$, edge set

$K(A,B) = (A \cup B, \{ab : a \neq b \wedge (a,b) \in A \times B\})$, graph

$K(A) = K(A,A)$, graph

$E_G(A,B) = \{ab \in E(G) : a \neq b \wedge (a,b) \in A \times B\}$, edge set

A.9 Matching card trick

Consider a deck of $2k+1$ cards numbered $1,\ldots,2k+1$, and denoted by $[2k+1]$. The class chooses a *hand* H consisting of $k+1$ of these cards, and gives them to Professor A. Professor A looks at them, puts one of them in his pocket, and then has a student spread the remaining k cards face-up on a table. Professor B, who has observed none of this transaction, now enters the room, looks at the cards on the table and identifies the one in Professor A's pocket. How is this done?

Solution. Our arithmetic is done modulo $k+1$, and we use $k+1$ instead of 0 for the representative of its equivalence class. Arrange the cards of H in order as $c_1 < \cdots < c_{k+1}$. Let $x = \sum_{c \in H} c \mod k+1$. Professor A hides card c_x. When Professor B arrives, he sees that the cards $d_1 < \cdots < d_{k+1}$ in $[2k+1] \setminus (H - c_x)$ are missing, and he calculates $y := \sum_{c \in H-x} = x - c_x$. The class is holding $c_x - 1 - (x-1) = -y$ cards less than c_x and Professor A is holding c_x. It follows that $c_x = d_{1-y}$, and Professor B can calculate the rhs.

Another way of saying this is that Professor B knows the missing cards $\overline{d}_1 > \cdots > \overline{d}_{k+1}$. Then $c_x = \overline{d}_{k+2-(1-y)} = \overline{d}_y$.

A.10 Old Versions of Proofs

A.10.1 Vizing's Theorem

Old Proof. Set $\Delta = \Delta(G)$. For each vertex v and $(\Delta+1)$-edge-coloring f, define $\overline{f}(v) = [\Delta+1] \smallsetminus \{f(e) : e \in E(v)\}$; then $\overline{f}(v)$ is the set of colors that v does not "see".

We argue by induction on $|G|$. The base step $|G| = 1$ is trivial, so consider the induction step $|G| > 1$. Fix $v \in V$. For $S \subseteq E(v)$, let $\overline{S} = E_G(v) \smallsetminus S$ and $G_S = G - \overline{S}$. By the induction hypothesis $G_\emptyset$ has a $(\Delta+1)$-edge-coloring $f_\emptyset$. Now supose that f_S is a $(\Delta+1)$-edge-coloring of G_S. Set $C_S(vw) = \overline{f}_S(w) \smallsetminus f_S(S)$; then $C_S(vw)$ is the set of colors that can be used to color vw in an extension of f_S to a $(\Delta+1)$-coloring of G_{S+vw}. Notice that

$$\forall vw \in E(v),\ |C_\emptyset(vw)| \geq \Delta + 1 - d_{G_\emptyset}(w) - |\emptyset| \geq 2.$$

Choose the largest subset $S \subseteq E(v)$ such that there exists a $(\Delta+1)$-edge-coloring f_S of G_S and:

1. $|C_S(vw)| \geq 1$ for all $vw \in \overline{S}$ and
2. if $S \neq E(v)$ then there exists $vu_S \in \overline{S}$ such that $|C_S(vw)| \geq 2$ for all $vw \in \overline{S} - vu_S$.

This is possible, since $S = \emptyset$ is a candidate. We claim that $S = E_G(v)$, and so $G = G_{E(v)}$ has an $(\Delta+1)$-edge-coloring. Suppose, for a contradiction, $S \neq E(v)$. For each edge $vw \in \overline{S}$, choose $\beta_w, \beta'_w \in C_S(vw)$ such that $\beta_w \neq \beta'_w$ if $w \neq u_S$. Consider cases:

Case 0: $|\overline{S}| = 1$. Then we can extend f_S to a $(\Delta+1)$-edge-coloring $f_{E(v)}$ of $G = G_{E(v)}$ by setting $f_{E(v)}(vu_S) = \beta_{u_S}$.

Case 1: There exist $vw_S \in \overline{S} - vu_S$ such that $\beta_{u_S} \notin \{\beta_x, \beta'_x\}$ for all $x \notin \{u_S, w_S\}$. Then setting $S' = S + vu_S$, extending f_S to a $(\Delta+1)$-edge-coloring $f_{S'}$ of $G_{S'}$ by putting $f_S(vu) = \beta_{u_S}$, and setting $vu_{S'} = vw_S$, contradicts the maximality of S.

Case 2: There exists $vw_S \in \overline{S} - vu_S$ and $\beta \in \{\beta_{w_S}, \beta'_{w_S}\}$ such that $\beta \notin \{\beta_x, \beta'_x\}$ for all $x \neq w_S$. Then setting $S' = S + vw_S$, extending f_S to a $(\Delta+1)$-edge-coloring $f_{S'}$ of $G_{S'}$ by putting $f_S(vw_S) = \beta$, and setting $vu_{S'} = vu_S$ contradicts the maximality of S.

Case 3: None of Cases 0–2 hold. Our plan is to modify f_S so that Case 1 will hold. Since Case 0 does not hold, $|\overline{S}| \geq 2$. Set $\beta = \beta_{u_S}$. Let M be the multiset $(\beta_w, \beta'_w : w \in \overline{S})$ and C be the associated set $\{\beta_w, \beta'_w : w \in \overline{S}\}$. Then $|M| = 2|\overline{S}| - 1$. Since Cases 1,2 do not hold, β appears at least three times in M, and every other color $\gamma \in C$ appears at least twice in M. It follows that $|C| \leq |\overline{S}| - 1$, and thus

$$\Delta + 1 - |C| - |f_S(S)| \geq \Delta + 1 - |\overline{S}| + 1 - |S| \geq \Delta + 2 - (|\overline{S}| + |S|) \geq 2.$$

So there exists a color $\alpha \in [\Delta+1] \smallsetminus (C \cup f_S(S))$.

Let F be the set of all edges colored α or β, and let P be the component of $G[F]$ containing u_S. Then P is a path starting at u_S and ending at another vertex z. Since $\alpha, \beta \notin f_S(S)$, $v \notin V(P)$. Let g be the $(\Delta+1)$-edge-coloring resulting from switching α and β. This does not change the color of any edge in S. It does put $\alpha \in C_S(vu_S)$, and $\alpha \in C_S(vz)$, if $vz \in \overline{S}$ and $f_S(z'z) = \alpha$, where $z'z$ is the last edge of P, but the other $C_S(vw)$ are unchanged. So changing β_{u_S} to α and possibly changing one of β_z, β'_z to α, leaves us in Case 1. This completes the contradiction. □

A.10.2 Galvin's Theorem

Lemma 122. *Let $D = (V, A)$ be a digraph such that every induced subgraph of D has a kernel. If f is a list assignment for D such that every vertex v satisfies $d^+(v) < |f(v)|$ then D has an f-coloring.*

Proof. We argue by induction on $|D|$.

Base Step: $|D| = 1$. Let v be the only vertex of D. Then $|f(v)| > d^+(v) = 0$. Thus there exists a color $\alpha \in f(v)$, and we can color v with α.

Induction Step: $|D| > 1$. Let $v_0 \in V$ and $\alpha \in f(v_0)$ (it exists as above). Let $W = \{v \in V : \alpha \in f(v)\}$. Note that $W \neq \emptyset$, since $v \in W$. By hypothesis $D[W]$ has a (nonempty) kernel S. Color every vertex in S with α. This is possible because S is independent and $\alpha \in f(v)$ for every vertex in S.

To finish, we extend our f-coloring to $D' = D - S$ so that α is not used for any vertex in $D - S$. For this purpose, let f' be the list assignment for D' defined by $f'(v) = f(v) - \alpha$. If D' has no vertices then we are done. Regardless, $|D'| < |D|$, so we are done by the induction hypothesis provided

$$|f'(v)| > d^+_{D'}(v) \text{ for all } v \in V \smallsetminus S.$$

If $v \notin W$ then $\alpha \notin f(v)$, and so

$$|f'(v)| = |f(v)| > d_D(v) \geq d_{D'}(v).$$

Otherwise, $v \in W$. Since S is a kernel for $D[W]$, there exists $w \in S \subseteq V \smallsetminus V(D')$ such that $vw \in A$. So

$$|f'(v)| = |f(v) - \alpha| > d^+_D(v) - 1 \geq d^+_{D'}(v).$$

□

Theorem 123. *Every X,Y-bigraph G satisfies $\chi'_l(G) = \Delta(G)$.*

Proof. Let $\Delta = \Delta(G)$ and set $H = L(G)$. Then $\chi'_l(G) = \chi_l(H) = \Delta$. Let $c : E(G) \to [\Delta]$ be a Δ-edge coloring of G. We will apply Lemma 122 to the following orientation $D = (E(G), A)$ of H. Note that for every edge $ee' \in E(H)$, either $e \cap e' \subseteq X$ or $ee' \subseteq Y$. If $e \cap e' \in X$, let $ee' \in A$ iff $c(e) > c(e')$; if $e \cap e' \in Y$ let $ee' \in A$ iff $c(e) < c(e')$. In order to apply Lemma 122, we need the following two claims.

Claim (A). $\Delta^+(D) \leq \Delta - 1$.

Proof. Every edge $e \in E(G)$ has at most $c(e) - 1$ out-neighbors e' with $e \cap e' \in X$ and at most $\Delta - c(e)$ out-neighbors e' with $e \cap e' \in Y$. $\square$

Claim (B). Every induced subgraph of $D' \subseteq D$ has a kernel.

Proof. Argue by induction on $|D'|$. The base step $|D'| = 1$ is trivial, so consider the induction step $|D'| > 1$. Suppose $D' = D[F]$, where $F \subseteq E(G) = V(D)$. Let $X' = \{x \in X : E_G(x) \cap F \neq \emptyset\}$. For each $x \in X'$, let e_x be the edge of $E_G(x) \cap F$ with the smallest color $c(e_x)$. Then for every $e = xy \in F$, we have $ee_x \in A$. Thus $Q = \{e_x : x \in X'\}$ is a kernel of D' if it is independent. Otherwise there exist distinct $x, x' \in X'$ such that $e_x \cap e_{x'} \neq 0$. So there exists $y \in Y$ with $e_x = xy$ and $e_{x'} = x'y$. Suppose $c(e_x) < c(e_{x'})$. Then $e_x e_{x'} \in A$.

Let $D'' = D' - e_x$. By the induction hypothesis, D'' has a kernel S. If $e_{x'} \in S$ then S is also a kernel for D', since $e_x e_{x'} \in A$. Otherwise, there exists $e^* \in S$ such that $e_{x'} e^* \in A$. Thus $e_{x'} \cap e^* \notin X$ by the definition of A and the choice of e'_x. So $e_{x'} \cap e^* \in Y$, and thus $e^* = x^* y$ for some $x^* \in X'$. By the definition of A, $c(e_{x'}) < c(e^*)$. Thus $e_x \cap e^* \in Y$ and $c(e_x) < c(e_{x'}) < c(e^*)$. So $e_x e^* \in A$. Thus S is also a kernel for D'. $\square$

Finally, consider any Δ-edge-list assignment f for G. It is also a Δ-list assignment for D. By Lemma 122, using Claims A and B, D has an f-coloring g, and g is also an f-edge-coloring of G. $\square$

Appendix B: Complex Networks

In the context of *network theory*, a *complex network* is a *graph* (network) with non-trivial topological features — features that do not occur in simple networks such as lattices or random graphs but often occur in graphs modelling of real systems.

Most social, biological, and technological networks display substantial non-trivial topological features, with patterns of connection between their elements that are neither purely regular nor purely random. Such features include a heavy tail in the *Degree Distribution*, a high *Clustering Coefficient*, *assortativity* or *disassortativity* among vertices, *Community Structure*, and *hierarchical structure*. In the case of directed networks these features also include reciprocity, triad significance profile and other features. In contrast, many of the mathematical models of networks that have been studied in the past, such as lattices and *random graphs*, do not show these features. The most complex structures can be realized by networks with a medium number of interactions [34]. This corresponds to the fact that the maximum information content (*entropy*) is obtained for medium probabilities.

Two well-known and much studied classes of complex networks are *Scale Free Networks* [8] and *Small World Networks*, [54] [6] whose discovery and definition are canonical case-studies in the field. Both are characterized by specific structural features — *power-law degree distributions* for the former and short *path lengths* and high clustering for the latter.

However, as the study of complex networks has continued to grow in importance and popularity, many other aspects of network structure have attracted attention as well.

B.1 Degree Distribution

The *degree* of a node in a network is the number of *connections* or *edges* the node has to other nodes. If a network is *directed*, meaning that edges point in one direction from one node to another node, then nodes have two different degrees, the *in-degree*, which is the number of incoming edges, and the *out-degree*, which is the number of outgoing edges.

The degree distribution $P(k)$ of a network is then defined to be the fraction of nodes in the network with degree k. Thus if there are n nodes in total in a network and n_k of them have degree k, then $P(k) = \frac{n_k}{n}$.

The same information is also sometimes presented in the form of a *cumulative degree distribution*, the fraction of nodes with degree smaller than k, or even the complimentary cumulative degree distribution, the fraction of nodes with degree greater than or equal to k ($1-C$ if one considers C as the cumulative degree distribution; i.e. the complement of C).

The degree distribution is very important in studying both real and theoretical networks. The simplest network model, for example, the (*Bernoulli*) *random graph*, in which each of n nodes is connected (or not) with independent probability p (or $1-p$), has a binomial distribution of degrees k:

$$P(k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

(or *Poisson* in the limit of large n). Most networks in the real world, however, have degree distributions very different from this. Most are highly right-skewed, meaning that a large majority of nodes have low degree but a small number, known as "hubs", have high degree. Some networks, notably the Internet, the world wide web, and some social networks are found to have degree distributions that approximately follow a power law: $P(k) \sim k^{-\gamma}$, where γ is a constant. Such networks are called *Scale Free Networks* and have attracted particular attention for their structural and dynamical properties.

B.2 Small World Networks

A *small-world network* is a type of mathematical graph in which most nodes are not neighbors of one another, but the neighbors of any given node are likely to be neighbors of each other and most nodes can be reached from every other node by a small number of hops or steps. Specifically, a small-world network is defined to be a network where the typical distance L between two randomly chosen nodes (the number of steps required) grows proportionally to the logarithm of the number of nodes N in the network, that is [54]

$$L \propto \log N$$

while the clustering coefficient is not small. In the context of a social network, this results in the small world phenomenon of strangers being linked by a short chain of acquaintances. Many empirical graphs show the small-world effect, e.g., social networks, the underlying architecture of the Internet, wikis such as Wikipedia, and gene networks.

Intuitively, a network is called a small-world network by analogy with the small-world phenomenon (popularly known as six degrees of separation). The small world hypothesis, which was first described by the Hungarian writer Frigyes Karinthy in 1929, and tested experimentally by Stanley Milgram (1967), is the idea that two arbitrary people are connected by only six degrees of separation, i.e. the diameter of the corresponding graph of social connections is not much larger than six. In 1998, Duncan J. Watts and Steven Strogatz published the first small-world network model, which through a single parameter smoothly interpolates between a random graph and a lattice. Their model demonstrated that with the addition of only a small number of long-range links, a regular graph, in which the diameter is proportional to the size of the network, can be transformed into a "small world" in which the average number of edges between any two vertices is very small (mathematically, it should grow as the logarithm of the size of the network), while the clustering coefficient stays large. It is known that a wide variety of abstract graphs exhibit the small-world property, e.g., random graphs and scale-free networks. Further, real world networks such as the World Wide Web and the metabolic network also exhibit this property.

For the purpose of this study, we tested the network as a whole for the presence of small world. If the entire network was not found to be a small world, the network was broken into graph connected components and iteratively tested for small world network, based on the small world score as given by Humphries et al. [30]. The test is based on a metric called *small world score*. We generate a random E–R graph with a given edge probability and define:

L_g: mean shortest path length of graph G

C_g^Δ: Clustering coefficient of G

L_{rand}: mean shortest path of random E-R graph

C_{rand}^Δ: Clustering coefficient of random E-R graph

where the definition of clustering coefficient, based on transitivity is given by:

$$C^\Delta = \frac{3 \times \text{number of triangles}}{\text{number of paths of length 2}}$$

with triangle being the set of three nodes, with each node connected to the other two.

A network in the analysis was considered small world if $L_g > L_{rand}$ and $C_g^\Delta \gg C_{rand}^\Delta$.

We then calculate the ratio of clustering coefficient of G to that of the random E-R graph,

$$\gamma_g^{\Delta} = \frac{C_g^{\Delta}}{C_{rand}^{\Delta}}$$

and the ratio of shortest path length of graph G to that of random E-R graph,

$$\lambda_g = \frac{L_g}{L_{rand}}$$

Small world score, S^{Δ} is then calculated as

$$S^{\Delta} = \frac{\gamma_g^{\Delta}}{\lambda_g}$$

A network qualifies to be a small world, if $S^{\Delta} > 1$.

B.3 Community Structure

Community structure refers to the occurrence of groups of nodes in a network that are more densely connected internally than with the rest of the network. This inhomogeneity of connections suggests that the network has certain natural divisions within it.

Communities are often defined in terms of the *partition* of the set of vertices, that is each node is put into one and only one community. This is a useful simplification and most community detection methods find this type of community structure. Social networks might have overlapping communities, where each vertex represents a person, and the communities represent the different groups of friends: one community for family, another community for co-workers, one for friends in the same sports club, and so on. The use of cliques for community detection is an example of how such overlapping community structure can be found.

Finding communities within an arbitrary network can be a computationally difficult task. The number of communities, if any, within the network is typically unknown and the communities are often of unequal size and/or density. Despite these difficulties, however, several methods for community finding have been developed and employed with varying levels of success.

Out of the available methods, we used the *Louvain method* [13], based on *modularity maximization*.

Modularity is a scale value (ranging between -1 and 1) that relates the structure of the community with the network outside the community by comparing the density of the edges within to those present outside the community. The Louvain Method is a greedy optimization method (complexity $O(nlogn)$), iteratively optimizing local communities until global modularity can no longer be improved, given small perturbations in community state.This method is an unsupervised, two-phase iterative method.

The value to be optimized is modularity, defined as a value between -1 and 1 that measures the density of links inside communities compared to links between communities [13]. For a weighted graph, modularity is defined as:

$$Q = \frac{1}{2m} \Sigma_{ij} \left[A_{ij} - \frac{k_i k_j}{2m} \right] \delta(c_i, c_j),$$

where: A_{ij} represents the edge weight between nodes i and j;
k_i and k_j are the sum of the weights of the edges attached to nodes i and j respectively;
m is the sum of all of the edge weights in the graph;
c_i and c_j are the communities of the nodes; and
δ is a simple delta function.

In order to maximize this value efficiently, the Louvain Method has two phases that are repeated iteratively.

First, each node in the network is assigned to its own community. Then for each node i, the change in modularity is calculated for removing i from its own community and moving it into the community of each neighbor j of i This value is easily calculated by [13]

$$\Delta Q = \left[\frac{\Sigma_{in} + 2k_{i,in}}{2m} - \left(\frac{\Sigma_{tot} + k_i}{2m}\right)^2\right] - \left[\frac{\Sigma_{in}}{2m} - \left(\frac{\Sigma_{tot}}{2m}\right)^2 - \left(\frac{k_i}{2m}\right)^2\right]$$

Where Σ_{in} is sum of all the weights of the links inside the community i is moving into, Σ_{tot} is the sum of all the weights of the links to nodes in the community, k_i is the weighted degree of i, $k_{i,in}$ is the sum of the weights of the links between i and other nodes in the community, and m is the sum of the weights of all links in the network. Then, once this value is calculated for all communities i is connected to, i is placed into the community that resulted in the greatest modularity increase. If no increase is possible, i remains in its original community. This process is applied repeatedly and sequentially to all nodes until no modularity increase can occur. Once this local maximum of modularity is hit, the first phase has ended.

In the second phase of the algorithm, it groups all of the nodes in the same community and builds a new network where nodes are the communities from the previous phase. Any links between nodes of the same community are now represented by self loops on the new community node and links from multiple nodes in the same community to a node in a different community are represented by weighted edges between communities. Once the new network is created, the second phase has ended and the first phase can be re-applied to the new network.

B.4 Clustering Coefficient

An important property of a graph is the clustering coefficient. The clustering coefficient is the measure of the the extent to which one's friends are also friends of each other. This measure has become popular due to a 1998 paper in Nature by Watts and Strogatz [54]. This property is sometimes called the local clustering coefficient.

A graph $G = (V,E)$ formally consists of a set of vertices V and a set of edges E between them. An edge e_{ij} connects vertex v_i with vertex v_j

The neighbourhood N_i for a vertex v_i is defined as its immediately connected neighbours as follows:

$$N_i = \{v_j : e_{ij} \in E \vee e_{ji} \in E\}.$$

The local clustering coefficient C_i for a vertex v_i is then given by the proportion of links between the vertices within its neighbourhood divided by the number of links that could possibly exist between them. For a directed graph, e_{ij} is distinct from e_{ji}, and therefore for each neighbourhood N_i there are $k_i(k_i - 1)$ links that could exist among the vertices within the neighbourhood (k_i is the number of neighbours of a vertex). Thus, the local clustering coefficient for directed graphs is given as [54]

$$C_i = \frac{|\{e_{jk} : v_j, v_k \in N_i, e_{jk} \in E\}|}{k_i(k_i - 1)}.$$

For an undirected graph, e_{ij} and e_{ji} are identical. Therefore if a vertex v_i has k_i neighbors, $\frac{k_i(k_i-1)}{2}$ edges could exist among the vertices within the neighbourhood. Therefore a local clustering coefficient for *undirected graphs* can also be defined as

$$C_i = \frac{2|\{e_{jk} : v_j, v_k \in N_i, e_{jk} \in E\}|}{k_i(k_i - 1)}.$$

A few more variations of methods for calculating clustering coefficients exist in literature. For unweighted graphs, the clustering of a node u is the fraction of possible triangles through that node that exist,

$$c_u = \frac{2T(u)}{deg(u)(deg(u)-1)},$$

where $T(u)$ is the number of triangles through node u and $deg(u)$ is the degree of u.

For weighted graphs, the clustering is defined as the geometric average of the subgraph edge weights [47],

$$c_u = \frac{1}{deg(u)(deg(u)-1))} \sum_{uv} (\hat{w}_{uv}\hat{w}_{uw}\hat{w}_{vw})^{1/3}.$$

The edge weights $\hat{w}_{uv}$ are normalized by the maximum weight in the network $\hat{w}_{uv} = w_{uv}/\max(w)$. The value of c_u is assigned to 0 if $deg(u) < 2$.

B.5 Scale Free Networks

A *scale-free network* is a network whose *degree distribution* follows a *power law*, at least asymptotically. That is, the fraction $P(k)$ of nodes in the network having k connections to other nodes goes for large values of k as

$$P(k) \sim k^{-\gamma}$$

where γ is a parameter whose value is typically in the range $2 < \gamma < 3$, although occasionally it may lie outside these bounds [43] [18].

Many networks have been reported to be scale-free, although statistical analysis has refuted many of these claims and seriously questioned others [20]. Preferential attachment and the fitness model have been proposed as mechanisms to explain conjectured power law degree distributions in real networks.

Intuitively, a network is named scale-free [8] if its degree distribution, i.e., the probability that a node selected uniformly at random has a certain number of links (degree), follows a particular mathematical function called a power law. The power law implies that the degree distribution of these networks has no characteristic scale. In contrast, networks with a single well-defined scale are somewhat similar to a lattice in that every node has (roughly) the same degree. Examples of networks with a single scale include the Erdős–Rényi (ER) random graph and hypercubes. In a network with a scale-free degree distribution, some vertices have a degree that is orders of magnitude larger than the average - these vertices are often called "hubs", although this is a bit misleading as there is no inherent threshold above which a node can be viewed as a hub. If there were such a threshold, the network would not be scale-free.

For the current study, we tested the presence of scale free degree distributions using the metric given by Clauset et al. [20].