CSU200 Discrete Structures   Professor Fell   Sequences and Sums

A *sequence* is a function *t* from a subset of the integers (usually N or Z^+) to a set S. In this discussion, the set S will be a set of numbers but we could have a sequence of colors, musical notes, or even computer programs.

$a_n = t(n)$. a_n is a *term* of the sequence.

The sequence $a_1, a_2, a_3, \cdots$ may be denoted by $\{ a_n \}$ or $\{ a_n \}_{n = 1..\infty}$. The numbers, 1, 2, 3, ... are the term indices and $a_1, a_2, a_3, \cdots$ are the terms or term values.

Examples:
Can you give the next term of each of these sequences? Can you the function *t(n)*that defines the sequence? Assume the first term is a_1.
1, 1/2 , 1/3, 1/4, 1/5, 1/6, ...
2, 4, 6, 8, 10, 12, ...
2, 4, 8, 16, 32, 64, ...
7, 8, 9, 10, 11, 12, ...
1/2 , 1, 2, 4, 8, 16, ...
1, 3, 6, 10, 15, 21, ...
1, 1, 1, 1, 1, 1, 1, ...
1, 1, 2, 3, 5, 8, ...
1/2, -2/3, 3/4, -4/5, 5/6, -6/7, ...
2, 3, 5, 7, 11, 13, ...

Some Special Sequences:
Arithmetic Sequence
An *arithmetic sequence* is a sequence in which each term equals the preceding term plus a constant. A general arithmetic sequence looks like this $a, a + d, a + 2d, a + 3d, \ldots$
The first term is a and the constant difference between terms is d.
What is the *nth* term of the sequence $a, a + d, a + 2d, a + 3d, \ldots$?

What is the next term and the *nth* term of each of these sequences?
4, 7, 10, 13, 16, 19, ...
-7, -1, 5, 11, 17, 23, ...

Geometric Sequence
A *geometric sequence* is a sequence in which each term equals the preceding term times a constant. A general geometric sequence looks like this $a, ar, ar^2, ar^3, ar^4, \ldots$
he first term is a and the constant ratio between successive terms is r.
What is the *nth* term of the sequence $a, ar, ar^2, ar^3, ar^4, \ldots$?

What is the next term and the *nth* term of each of these sequences?
2, 6, 18, 54, 162, . . .
1, -4, 16, -64, 256, . . .

Quadratic Sequence

A *quadratic sequence* is a sequence whose *nth* term is given by a quadratic function, $a_n = an^2 + bn + c$.

Here are some quadratic sequences. Can you find the quadratic function that generates them?

1, 4, 9, 16, 25, 36, ...

6, 15, 28, 45, 66, 91, ...

An arithmetic sequence is given by a linear function and the differences between successive terms is a constant. In a quadratic sequence the differences between successive terms is given by a linear function and the second differences are constant. For example:

sequence	6,		15,		28,		45,		66,		91, ...
differences		9,		13,		17,		21,		25, ...	
second differences			4,		4,		4,		4, ...		

Series and Partial Sums

A *series* is a sum of the terms of a sequence. Since a sequence has infinitely many terms, a series is the sum of infinitely many terms. We often sum only the first *n* terms of a sequence. The sum of the first *n* terms is called the *nth* partial sum.

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots + a_n + \cdots$$

This is a sum of an infinite number of terms.

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$ This is the *nth partial sum*.

k is the *index of summation; 1* is the *lower limit; n* is the *upper limit.*

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_n$$ This is also a partial sum.

j is the *index of summation; m* is the *lower limit; n* is the *upper limit.*

Arithmetic Series

An *arithmetic series* is a sum of terms of an arithmetic sequence.

$$\sum_{k=1}^{n} (a + dk) = (a+d) + (a+2d) + \cdots + (a+nd) = na + (1 + 2 + \cdots + n)d$$

$$= na + \frac{n(n+1)}{2}d = n\left(\frac{a+a+n+1}{2}\right) = n\left(\frac{\text{first term + last term}}{2}\right).$$

Examples

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$1+2+\cdots+100=\frac{100(100+1)}{2}=5050$$

$$2+5+8+11+14+17+20+23+26+29=\frac{10(31)}{2}=155$$

Geometric Series

A *geometric series* is a sum of terms of a geometric sequence.

$$\sum_{k=1}^{n} ar^k = ar+ar^2+ar^3+\cdots+ar^n$$

or, summing from 0,

$$\sum_{k=0}^{n} ar^k = a+ar+ar^2+ar^3+\cdots+ar^n$$

$$\sum_{k=0}^{\infty} ar^k = a+ar+ar^2+ar^3+\cdots$$ an infinite geometric series.

Let's find a formula the *nth* partial sum $\sum_{k=0}^{n} ar^k = a+ar+ar^2+ar^3+\cdots+ar^n$.

Let S = $\sum_{k=0}^{n} ar^k = a+ar+ar^2+ar^3+\cdots+ar^n$.

Then rS = $\sum_{k=0}^{n} ar^{k+1} = ar+ar^2+ar^3+\cdots+ar^n+ar^{n+1}$.

$S-rS=a-ar^{n+1}$ as all the other terms cancel out when we subtract rS from S.

So $S=\frac{a-ar^{n+1}}{1-r}=\frac{a\left(1-r^{n+1}\right)}{1-r}$.

Examples

$$\sum_{k=0}^{6}\frac{1}{3^k}=1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\frac{1}{3^5}+\frac{1}{3^6}=\frac{1\left(1-\frac{1}{3^7}\right)}{1-\frac{1}{3}}=\frac{3^7-1}{3^7-3^6}=\frac{2187\text{-}1}{2187\text{-}729}=\frac{2186}{1458}\approx 1.499$$

$$\sum_{k=0}^{6}3^k=1+3+3^2+3^3+3^4+3^5+3^6=\frac{1\left(1-3^7\right)}{1-3}=\frac{3^7-1}{2}=\frac{2187\text{-}1}{2}=\frac{2186}{2}=1093$$

$$\sum_{k=0}^{6}\frac{1}{2^k}=1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}=\frac{1\left(1-\frac{1}{2^7}\right)}{1-\frac{1}{2}}=\frac{2^7-1}{2^7-2^6}=\frac{128\text{-}1}{128\text{-}64}=\frac{127}{64}\approx 1.984$$

$$\sum_{k=0}^{n}\frac{1}{2^k}=\frac{1\left(1-\frac{1}{2^{n+1}}\right)}{1-\frac{1}{2}}=\frac{1-\frac{1}{2^{n+1}}}{\frac{1}{2}}=2-\frac{1}{2^n}$$ What happens when n tends to ∞?

What is $\sum_{k=0}^{\infty}\frac{1}{2^k}$?

Some Special Sums:

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n}k^3=\frac{n^2(n+1)^2}{4}$$