CSU200 Discrete Structures Professor Fell Polynomials

A *polynomial* is an expression that is a sum of powers of one or more variables multiplied by *coefficients*. The coefficients are numbers, e.g. integers, rationals, reals, or complex numbers. The pieces of the polynomial that are added or subtracted are called *terms*.

Examples:

$$x^7 + y^7 - xyz + 5$$
$$x - 14$$
$$z^2 - 5z + 6$$
$$3$$

For the rest of this document, **polynomial** will mean a polynomial in one variable, like the last three polynomials above. The *degree* of a polynomial is the highest power of the variable that occurs with a non-zero coefficient. By definition, a constant polynomial is said to have *degree 0*.

$x - 14$	degree 1
$z^2 - 5z + 6$	degree 2
3	degree 0

We have special names for the lower degree polynomials.

Degree	**Name**
0	*constant*
1	*linear*
2	*quadratic*
3	*cubic*
4	*quartic*
5	*quintic*

We sometimes use literals for the coefficients when we are talking about a family of polynomials.

Examples:

$ax^2 + bx + c$	a general quadratic polynomial
$5x^3 + 2x^2 + tx + 7$	a different cubic polynomial for each value of t
$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$	a general polynomial of *degree* less than or equal to n

Polynomials are functions that are defined for all real (or complex) values of their variable(s). We usually use *x* for the *input variable* and *y* for the polynomial value when we are working with real values. Three things we frequently do with polynomials are: evaluate them (compute their values), draw their graphs, and compute their *roots*, input values where the polynomial value is zero.

Example:

$$y = x^3 - 3x^2 - 4x + 12$$

When *x = 1*, y = 6. The value of the polynomial $y = x^3 - 3x^2 - 4x + 12$ is 6 at x = 1.
When *x = 0*, y = 12. The value of the polynomial $y = x^3 - 3x^2 - 4x + 12$ is 12 at x = 0.

The *roots* or zeros of the polynomial $y = x^3 - 3x^2 - 4x + 12$ are found by solving the polynomial $x^3 - 3x^2 - 4x + 12 = 0$. The roots are $x = 2$, $x = 3$, and $x = -2$.

Evaluating Polynomials

How do we compute the value of $y = x^3 - 3x^2 - 4x + 12$ at x = 5? We just substitute 5 for x and do the resulting arithmetic.

$\left(x^3 - 3x^2 - 4x + 12\right)\Big|_{x=5}$ means $x^3 - 3x^2 - 4x + 12$ evaluated at x = 5.

$$\left(x^3 - 3x^2 - 4x + 12\right)\Big|_{x=5} = 5^3 - 3\times 5^2 - 4\times 5 + 12 = 5\times 5\times 5 - 3\times 5\times 5 - 4\times 5 + 12$$

and

$$\begin{aligned} 1\times 5\times 5\times 5 - 3\times 5\times 5 - 4\times 5 + 12 &= 125 - 75 - 20 + 12 && \cdots\text{six multiplications} \\ &= 42 && \cdots\text{three additions or subtractions} \end{aligned}$$

Some questions about evaluating polynomials

Why did we count 1×5 as a multiplication when you really don't have to do anything to get the answer?

How many multiplications and how many additions or subtractions must you do to compute $\left(27x^4 + 5x^3 - 6x^2 + 43x + 23\right)\Big|_{x=97}$ in the same way we computed $\left(x^3 - 3x^2 - 4x + 12\right)\Big|_{x=5}$?

How many multiplications and how many subtractions of additions must you do to compute $\left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right)\Big|_{x=97}$ if you are given n and all the coefficients?

How many multiplications and how many subtractions of additions must you do to compute $\left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right)\Big|_{x=97}$ if *n = 100* and all the coefficients are non-zero? What is if *n = 1000*?

Horner's Method

Here is another way to evaluate $y = x^3 - 3x^2 - 4x + 12$.

$$y = x^3 - 3x^2 - 4x + 12 = 12 + x\left(x^2 - 3x - 4\right) = 12 + x\left(-4 + x\left(-3 + x(1)\right)\right)$$

and

$$\left(x^3 - 3x^2 - 4x + 12\right)\Big|_5 = 12 + 5\left(-4 + 5\left(-3 + 5(1)\right)\right)$$

This time we only did three multiplications and three additions or subtractions.

Why do we care?
Yes, we could write a computer program to do the work for us but it is still important to write programs that use efficient algorithms. Much of the work done on computers involves evaluating polynomials or other functions. The calculations might be time consuming even for a computer. Efficient algorithms can substantially cut the time whether it is for communicating with a Mars rover or getting out the next image for "*Traitor's Gate.*"

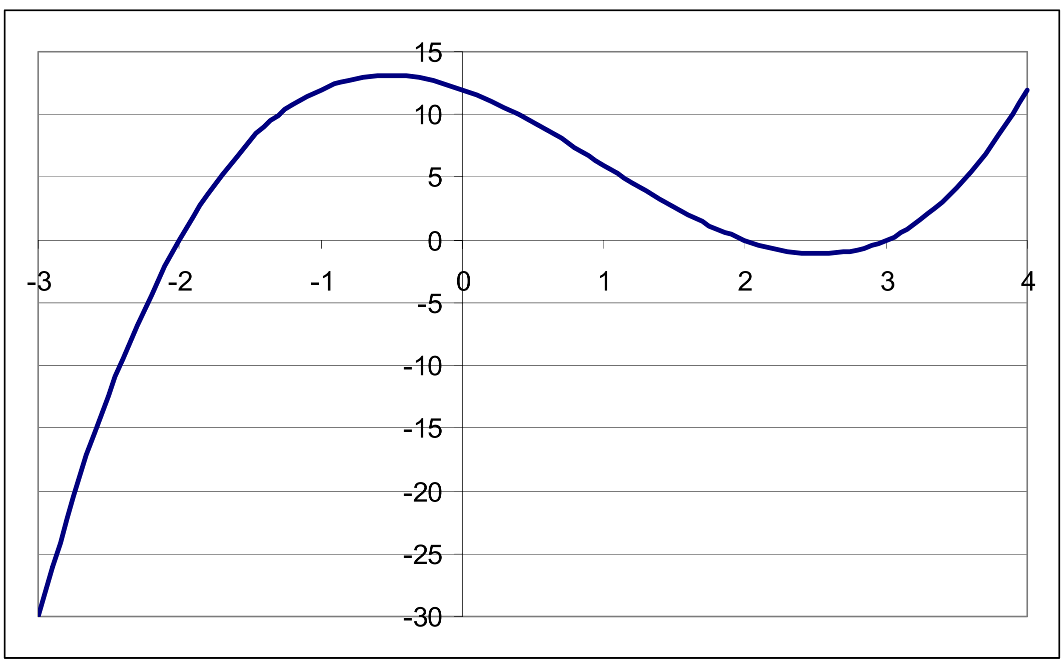
How many multiplications and how many additions or subtractions must you do to compute $27x^4 + 5x^3 - 6x^2 + 43x + 23$, for a particular *x,* using Horner's Method? That is, by using $\left(27x^4 + 5x^3 - 6x^2 + 43x + 23\right) = 23 + x\left(43 + x\left(-6 + x\left(5 + x\left(27\right)\right)\right)\right)$.

How many multiplications and how many subtractions of additions must you do to compute $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ using Horner's Method, if *x, n,* and all the coefficients are given?

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_0 + x\left(a_1 + x\left(a_2 + x(\cdots + x\left(a_n\right))\right)\right)$$

How many multiplications and how many subtractions of additions must you do to compute $\left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right)\Big|_{x=97}$ using Horner's Method, if $n = 100$ and all the coefficients are non-zero? What is if $n = 1000$?

Graphing Polynomials
As usual, when we plot the graph of a polynomial like $y = x^3 - 3x^2 - 4x + 12$, we use the horizontal axis for *x* values and the vertical axis for *y* values.



$y = x^3 - 3x^2 - 4x + 12$

Some questions about graphs of polynomials

What do the graphs of these polynomials look like?

$y = 3$ $\qquad$ $y = c$ where c is a real constant

$y = 5x - 7$ $\qquad$ $y = ax + b$ where a and b are real constants

$y = 2x^2 - 3x + 5$

$y = -2x^2 - 3x + 5$

$y = ax^2 + bx + c$ where a, b, and c are real constants

$y = ax^3 + bx^2 + cx + d$ where a, b, c, and d are real constants

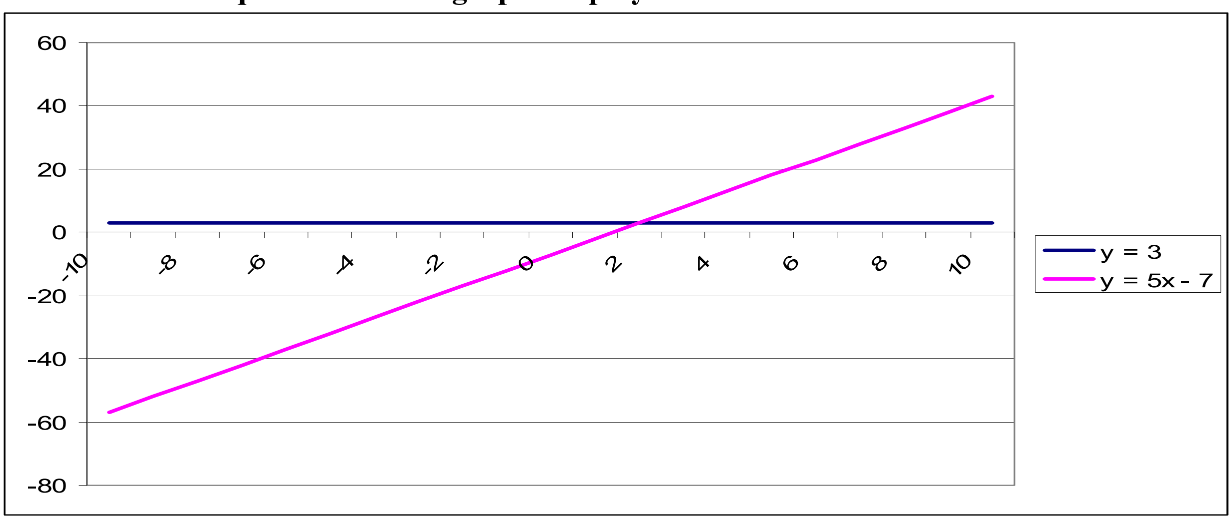
What can you say, in general, about the values and graphs of

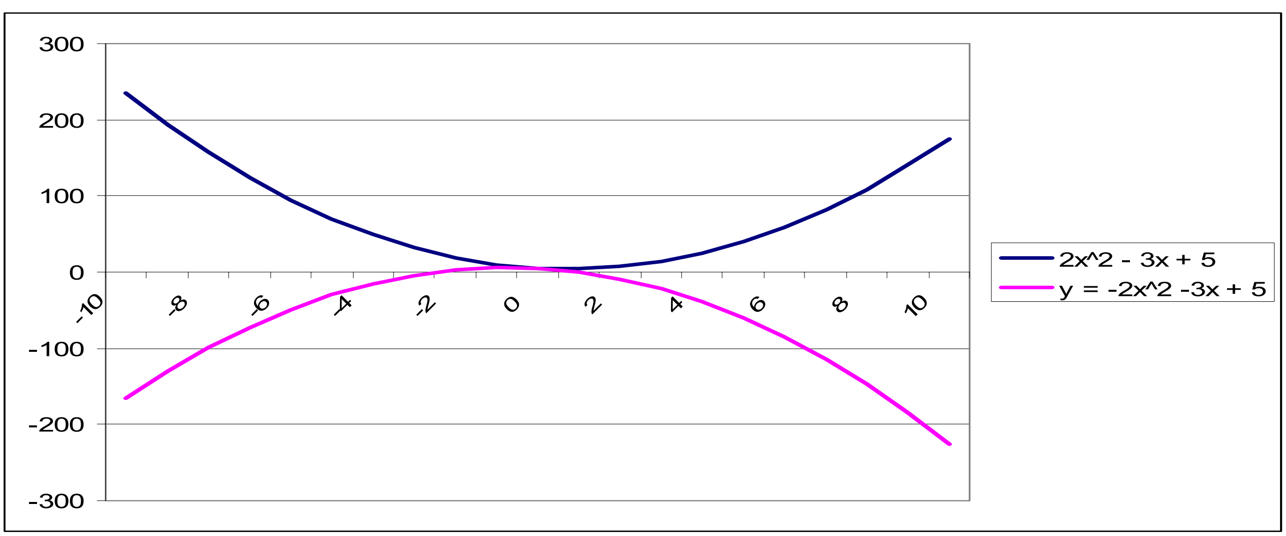
$$y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\,?$$

What is y when $x=0$, when $x=1$, when x tends to plus infinity, when x tends to minus infinity?

Which is bigger $\quad x^2 + x + 6 \quad$ or $\quad 10000x \quad$?

Some answers to questions about graphs of polynomials





What can you say, in general, about the values and graphs of

$$y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\ ?$$

What is y when x=0?	a_0
When x=1?	$a_n + a_{n-1} + \cdots + a_1 + a_0$
When x tends to plus infinity?	+∞ if $a_n > 0$, -∞ if $a_n < 0$
	depends on a_{n-1} if $a_n = 0$
When x tends to minus infinity?	-∞ if $a_n > 0$, +∞ if $a_n < 0$
	depends on a_{n-1} if $a_n = 0$

Which is bigger $x^2 + x + 6$ or $10000x$?

The answer depends on the value of x. Here are some values:

x	x*x+x+6	10000x
0	**6**	0
0.00001	**6.00003**	0.1
0.1	6.3	**1000**
1	9	**10000**
10	36	**100000**
100	306	**1000000**

Finding Roots of Polynomials

Finding the *root* of a polynomial means finding the solutions to the equation *y=0*, or

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0\,.$$

It is not always possible to give a formula for the solution of a polynomial equation. For example, unlike quadratic, cubic, and quartic polynomials, the *general* quintic cannot be solved algebraically in terms of a finite number of additions, subtractions, multiplications, divisions, and root extractions. This was proved by Abel and Galois.

There are method's for approximating the roots of arbitrary polynomials with great accuracy. You will probably learn one of these methods, Newton's Method, in your calculus course. Here, we'll just review the formulas for finding (*extracting*) the roots of polynomials of degree 0, 1, and 2.

$y = c$ has no solutions if $c \neq 0$. Every x is a solution if $c=0$.

$y = ax + b$, where $a \neq 0$, has exactly one root, $x = \frac{-b}{a}$.

$y = ax^2 + bx + c$, where $a \neq 0$, has roots given by the *quadratic formula*.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Some questions about roots of polynomials

Does the equation

$$ax^2 + bx + c = 0$$

have exactly two solutions whenever a, b, and c are real numbers?

Hint: What happens if $a = 0$?
What if a and b are both 0 and $c \neq 0$?
What if a, b and c are all 0?
How many solutions does $x^2 - 2x + 1 = 0$ have?
(The only solution is 1 but it occurs with multiplicity 2.)

What are the roots of $y = (2x-6)(5x-9)(x+1)(7x-1)(3x+12)$? Is this a polynomial?

Hint: Each factor gives a root. Yes, it is a polynomial of degree 5.

What are the roots of $x^2 + 121$?

Hint: They are imaginary.

For what values of b does $x^2 + bx + 1$ have two distinct real roots? For what value of b does it have only one real root?

Hint: Remember the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-b \pm \sqrt{b^2 - 4}}{2}$ gives the roots of the equation. It will have only one real root when $b^2 - 4$ is zero.

How many roots does $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

Hint: Look at the hint above for the question about the roots of $ax^2 + bx + c = 0$.

References:

MathWorld Polynomial Page
History of Solutions of Polynomial Equations
Evaluation of Polynomials
William George Horner