CSU200 Discrete Structures Professor Fell Field Axioms

There are eleven axioms that apply to real arithmetic. Together they are called the *field axioms*. Five of the axioms are about addition (+); five are about multiplication (×), and one is about the interaction of these two operations. We often write xy instead of $x \times y$ so the axioms that involve multiplication are written both ways.

Closure: The set of real numbers is closed under addition and multiplication. That is, if x and y are real numbers then

CL+ $x + y$ is a unique real number.

CL× $x \times y$ is a unique real number. xy is a unique real number.

Commutativity: Addition and multiplication are *commutative* operations on the real numbers. That is, if x and y are real numbers then

CO+ $x + y = y + x$

CO× $x \times y = y \times x$ $xy = yx$

Associativity: Addition and multiplication are *associative* operations on the real numbers. That is, if x, y, and z are real numbers then

A+ $x + (y + z) = (x + y) + z$

A× $x \times (y \times z) = (x \times y) \times z$ $x(yz) = (xy)z$

Distributivity: Multiplication distributes over addition. That is, if x, y, and z are real numbers then

D $x \times (y + z) = (x \times y) + (x \times z)$ $x(y + z) = (xy) + (xz) = xy + xz$

Identity Elements: The set of real numbers contains

ID+ a unique *identity element for addition*, 0.
That is $x + 0 = x$ for any real number x.

ID× a unique *identity element for multiplication*, 1.
That is $x \times 1 = x$ for any real number x.

Inverses: The set of real numbers contains

IV+ a unique *additive inverse* for every real number x.
That is every real number x has a $-x$ such that $x + (-x) = 0$.

IV× a unique *multiplicative inverse* for every non-zero real number x.
That is every non-zero real number x has a $\frac{1}{x}$ such that $x \times \frac{1}{x} = 1$.

The rational numbers $\mathbb{Q}$ and the complex numbers $\mathbb{C}$ are also fields. We will study other fields this semester. The field axioms stated for a general field follow.

Any set S with addition and multiplication operations that obeys all eleven axioms below is a *field.* We will use the symbols + and × to denote addition and multiplication operations defined on the set S but other symbols are possible, e.g. ⊕ is often used for addition and ⊗, •, ·, ∘, *, or no symbol for multiplication

Closure: The set of S is closed under addition and multiplication. That is, if x and y in S
CL+ $x + y$ is defined and in S.
CL× $x \times y$ is defined and in S.

Commutativity: Addition and multiplication are *commutative* operations on the real numbers. That is, if x and y are in S then
CO+ $x + y = y + x$
CO× $x \times y = y \times x$

Associativity: Addition and multiplication are *associative* operations on the real numbers. That is, if x, y, and z are in S then
A+ $x + (y + z) = (x + y) + z$
A× $x \times (y \times z) = (x \times y) \times z$

Distributivity: Multiplication distributes over addition. That is, if x, y, and z are in S then
D $x \times (y + z) = (x \times y) + (x \times z)$

Identity Elements: The set S contains
ID+ a unique *identity element for addition*, 0.
That is $x + 0 = x$ for any x in S.
ID× a unique *identity element for multiplication*, 1.
That is $x \times 1 = x$ for any x in S.

Inverses: The set S contains
IV+ a unique *additive inverse* for every x in S.
That is every x in S has a $-x$ in S such that $x + (-x) = 0$.
IV× a unique *multiplicative inverse* for every non-zero x in S.
That is every non-zero x in S has a $\frac{1}{x}$ in S such that $x \times \frac{1}{x} = 1$.