# CSU200 Discrete Structures Professor Fell Exponentials and Logs

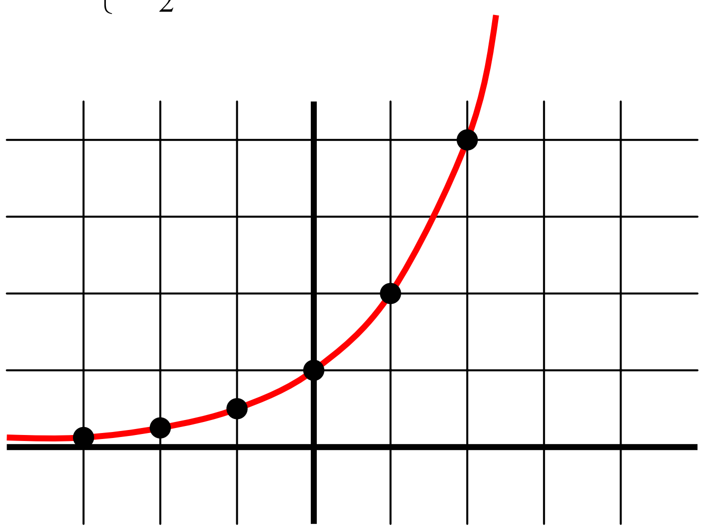
An *exponential* function has an equation of the form

$$y = a \cdot b^x \qquad b > 0$$

where $a$ is the constant of proportionality and $b$ is the *base*.

An exponential function $y = a \cdot b^x$ is defined for all real x. Let's look at the example, $y = 2^x$, before we state the general rules. (You have probably seen the function $y = (10)^x$ in high school and $y = e^x$ if you did AP calculus, but $y = 2^x$ is very important in computer science.) It is easy to define $2^x$ for integer values of $x$.

$$2^x = \begin{cases} \underbrace{2 \cdot 2 \cdots 2}_{x\ twos} & \text{if } x \text{ is a positive integer} \\ 1 & \text{if } x = 0 \\ \dfrac{1}{2^{-x}} & \text{if } x \text{ is a negaitive integer} \end{cases}.$$



***y=2^x* plotted at integer values of *x*, *-3 ≤ x ≤ 2* and extended smoothly.** Grid lines are at integer values.

We define $y = 2^x$ at all rational values of $x$ as follows. Set $x = \frac{m}{n}$ where $m$ and $n$ are integers and $n$ is positive, so the sign of $x$ is the same as the sign of $m$.

$$2^{\frac{m}{n}} = \left(\sqrt[n]{2}\right)^m$$

This agrees with the definition above when $x$ is an integer. These values fit along the graph we have already drawn.

Values of $y = 2^x$ can be defined rigorously at irrational values of *x* using limits of values at rational approximations to *x*. We won't be that rigorous here (It is material for a mathematical analysis course.) but if we want to compute $2^\pi$, for example, we know that we could approximate is as closely as we like by computing

$$2^3, \quad 2^{3.1}, \quad 2^{3.14}, \quad 2^{3.145}, \text{ and so on.}$$

**Properties of Exponentiation**

*Product of two powers with the same base*

$$b^x \cdot b^y = b^{x+y}$$

To multiply powers of the same base, add the exponents.

*Quotient of two powers with the same base*

$$\frac{b^x}{b^y} = b^{x-y}$$

To divide powers of the same base, subtract the exponent of the numerator from the exponent of the denominator.

*Power of a Power*

$$\left(b^x\right)^y = b^{xy}$$

To raise a power to a power, multiply the exponents.

*Power of a Product*

$$(xy)^a = x^a y^a$$

To raise a product to a power, raise each factor to that power.

*Power of a Quotient*

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

To raise a quotient to a power, raise the numerator and denominator to that power.

**Proving Things about Exponents**

If *b* is a positive real number and *n* is a positive integer then, as we did for *b = 2* above, we define

$$b^n = \underbrace{2 \cdot 2 \cdots 2}_{n \text{ twos}}.$$

Note: Later this semester, you will learn about recursion in CSU211. We can define $b^n$ recursively for positive integers *n* by

$$b^n = \begin{cases} b & \text{if } n = 1 \\ b \cdot b^{n-1} & \text{if } n > 1 \end{cases}.$$

Assume that $b$ is a positive real number and $n$ is a positive integer. Try using the first definition and only the properties of exponentiation listed above to prove:

$$b^0 = 1$$

$$b^{-n} = \frac{1}{b^n}$$

$$b^{\frac{1}{n}} = \sqrt[n]{b}$$

$$b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m$$

**Logarithms**

The *logarithm base b* is defined by

$$y = \log_b x \quad \text{means} \quad b^y = x.$$

The functions $\log_b x$ and $b^x$ are *inverse* functions which means:

If $y = b^x$, then $\log_b y = x$.

If $y = \log_b x$, then $b^y = x$.

We will talk more generally about relations, functions, and inverse functions later in the semester.

**Properties of Logarithms**

*Logarithm of a product*

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

To compute the logarithm of a product, add the logarithms of the factors.

*Logarithm of a quotient*

$$\log_b \frac{x}{y} = \log_b(x) - \log_b(y)$$

To compute the logarithm of a quotient, subtract the logarithm of the denominator from the logarithm of the numerator..

*Logarithm of a Power*

$$\log_b\left(x^c\right) = c\left(\log_b x\right)$$

To compute the logarithm of a power multiply the logarithm of the base of the argument by the exponent of the argument. ($x^c$ is the argument.)

**Some questions about logarithms**
Without a calculator, evaluate

$$\log_2(1) = \qquad \log_2(256) =$$
$$\log_2(2) = \qquad \log_2(1024) =$$
$$\log_2(4) = \qquad \log_2(.5) =$$
$$\log_2(32) = \qquad \frac{\log_2(9)}{\log_2\left(\frac{1}{9}\right)} =$$

Without a calculator, give approximate values for

$$\log_2(1000) \approx$$
$$\log_2(1{,}000{,}000) \approx$$

What does the graph of $\boldsymbol{y = \log_2 x}$ look like?

**Proving Things about Logarithms**
*Changing the base*

$$\log_a x = \frac{\log_b x}{\log_b a}$$

If *a* and *b* are both positive numbers then

$$(\log_a b)(\log_b a) = 1$$

**References:**
"Algebra and Trigonometry: Functions and Applications" by Paul A. Foerster, Chapter 6, Addison-Wesley, 1980.

"Discrete Mathematics, second edition" by James L. Hein, pages 85-87, Jones and Bartlett Mathematics, 2003.

**Proofs of statements about exponents**

$b^0 = 1$ $\qquad \dfrac{b^x}{b^y} = b^{x-y}$ so $b^0 = b^{n-n} = \dfrac{b^n}{b^n} = 1.$

$b^{-n} = \dfrac{1}{b^n}$ $\qquad$ Since $b^x \cdot b^y = b^{x+y}$ we have $b^n \cdot b^{-n} = b^{n+(-n)} = b^0 = 1$ which means $b^{-n} = \dfrac{1}{b^n}.$

$b^{\frac{1}{n}} = \sqrt[n]{b}$ $\qquad$ Since $\left(b^x\right)^y = b^{xy}$ we have $\left(b^{\frac{1}{n}}\right)^n = b^{\frac{1}{n}\cdot n} = b^1 = b.$

So by definition of the nth root, $b^{\frac{1}{n}} = \sqrt[n]{b}.$

$b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m$ Since $b^{\frac{1}{n}} = \sqrt[n]{b}$, $\left(\sqrt[n]{b}\right)^m = \left(b^{\frac{1}{n}}\right)^m = b^{\frac{m}{n}}$ by the Power of a Power rule.

**Answers to some questions about logarithms**

Without a calculator, evaluate

$\log_2(1) = 0$ $\qquad \log_2(256) = 8$

$\log_2(2) = 1$ $\qquad \log_2(1024) = 10$

$\log_2(4) = 2$ $\qquad \log_2(.5) = -1$

$\log_2(32) = 5$ $\qquad \dfrac{\log_2(9)}{\log_2\left(\frac{1}{9}\right)} = \dfrac{\log_2(9)}{-\log_2(9)} = -1$

Without a calculator, give approximate values for $\log_2(1{,}000)$ and $\log_2(1{,}000{,}000)$.

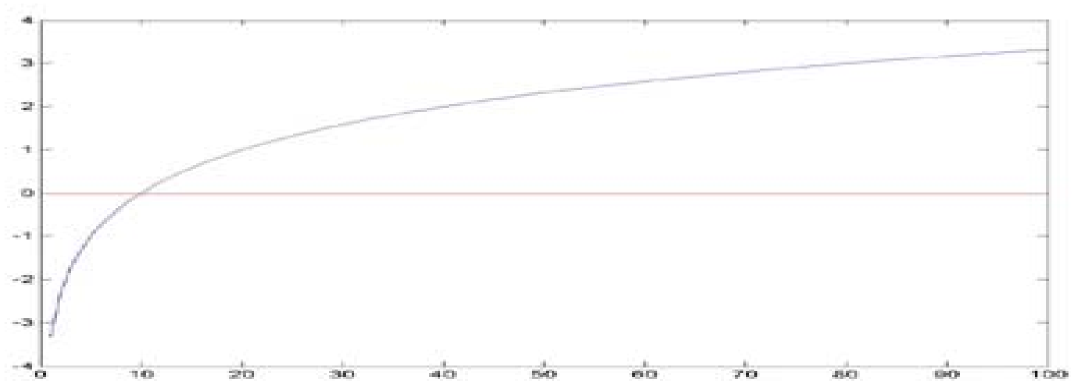
$\log_2(1000) \approx \log_2(1024) = 10$

According to my PC calcularor, $\log_2(1000) \approx 9.9657842846620870436109582884682$

so 10 really is a good approximation.

$\log_2(1{,}000{,}000) = \log_2(1{,}000 * 1{,}000) = \log_2(1{,}000) + \log_2(1{,}000) \approx 20$

What does the graph of $\boldsymbol{y = \log_2 x}$ look like?



**Proofs of Statements about Logarithms**
*Changing the base*

$$\log_a x = \frac{\log_b x}{\log_b a}$$

By the definition, $\log_a x$ is the unique number such that, $a^{\log_a x} = x$.

Similarly, $b^{\log_b x} = x$ and $b^{\log_b a} = a$.

Hence $a^{\frac{\log_b x}{\log_b a}} = \left(b^{\log_b a}\right)^{\frac{\log_b x}{\log_b a}} = b^{(\log_b a)\frac{\log_b x}{\log_b a}} = b^{\log_b x} = x$.

So $\dfrac{\log_b x}{\log_b a} = \log_a x$.

If $a$ and $b$ are both positive numbers then

$$(\log_a b)(\log_b a) = 1$$

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$a^{(\log_a b)(\log_b a)} = \left(a^{(\log_a b)}\right)^{(\log_b a)} = (b)^{(\log_b a)} = a$.

Hence $(\log_a b)(\log_b a) = \log_a a = 1$.