CSU200 Discrete Structures Professor Fell Counting

Ward Robe has 15 pairs of slacks and 5 shirts.
Natalie Attired 15 dresses and 5 pants suits.
(Names and attire based on problems in Foerster.)
How many different outfits can Ward choose from?
How many different outfits can Natalie choose from?

Though the questions look similar and even the numbers are the same, the answers are different.

Ward must choose one pair of slacks and one shirt to dress.

He has 9 ways to choose a pair of slacks and each pair of slacks can go with any of the five shirts for a total of 9*5 = 45 outfits.

Natalie must choose either a dress or a pants suit for a total of 9+5 = 14 outfits.

The two different counting rules that we used can be generalized. Ward's dressing problem generalizes to the Product Rule.

The Product Rule:

[Rosen, p. 302] Suppose a procedure can be broken down into a sequence of two tasks. If there are n_1 ways of doing the first task and n_2 ways of doing the second task after the first has been done, then there are n_1n_2 ways to do the procedure.
[Hein, p. 49] If A and B are finite sets then $|A \times B| = |A| \times |B|$.
[Fell] If A and B are finite sets then the number of ways of choosing an element from A and an element from B is $|A| \times |B|$.

Natalie's dressing problem generalizes to the Sum Rule.

The Sum Rule:

[Rosen, p. 305] If a first task can be done in n_1 ways and a second task in n_2 ways, and if these tasks cannot be done at the same time then there are $n_1 + n_2$ ways to do one of these tasks.
[Fell] If A and B are **disjoint** finite sets then the number of ways of choosing an element from A or B is $|A \cup B| = |A| + |B|$.

Examples:
Personal Identification Numbers or PINs are entered on a numeric keypad and, hence made up entirely of digits.

The PINs on our office locks are restricted to 4 digits. How many different PINs are possible?

The set of digits, D = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} has cardinality 10. Each PIN corresponds to an element of $D \times D \times D \times D$. There are 10^4 = 10,000 different PINs.

How many different 7 digit PINs are there? $10^7 = 10{,}000{,}000$
How many different 4 to 7 digit PINs are there.

A single PIN has either 4 or 5 or 6 or 7 digits. We use the product rule to separately count the sets of 4-digit , 5-digit, 6-digit, and 7-digit passwords then use the sum rule to count the union of these sets.

The number of 4 to 7 digit pins is $10^4 + 10^5 + 10^6 + 10^7 = 11{,}110{,}000$.

Passwords are often composed of alpha-numeric characters, {a, b, ... , z, 0, 1, 2, ... , 9} on systems that are not case-sensitive or {A, B, ... , Z, a, b, ... , z, 0, 1, 2, ... , 9} on systems that are case-sensitive.

How many 4-char alpha numeric passwords are there if you can use upper- and lower-case letters and digits (i.e. case-sensitive)?

There are 26 upper-case letters, 26 lower-case letters, and 10 digits for a character set C of size 62. The total number of possible passwords is

$$| C \times C \times C \times C| = (62)^4 = 14776336.$$

If a hacker has code that can try out passwords on a system at a rate of 1 per second, how long would it take her to break into a system that

a) uses 4-digit passwords?
10000 seconds = 2 hours 46 minutes 40 seconds.

b) uses 4-char case-sensitive, alpha-numeric passwords?
14776336 seconds = 171 days 32 minutes 16 seconds

Bitstrings are strings composed of 0s and 1s.
How many bit strings are there with 8 bits?

$2^{16} = 256$

How many bit strings are there with 16 bits?

$2^{16} = 65536$n

What is the largest integer that can be represented in 16-bit two's complement? MAXINT

Since positive integers in two's complement must have a 0 in the leftmost position, we have only 15 places to represent the magnitude of the integer. The largest integer we can represent is $111111111111111 = 2^{16} - 1 = 32767$.

If $A_1, A_2, \ldots, A_n$ are finite sets, what is the cardinality of $A_1 \times A_2 \times \ldots \times A_n$?

This is just a generalization of the product rule.

$$|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \times |A_2| \times \ldots \times |A_n|$$

Picking Students:
The three discrete structures sections have 73, 64 and 41 students.

a) How many distinct ways are there of choosing one discrete structures students to write up a sheet of notes for everyone to use at the final?

One student from the union of the three sections: 73 + 64 + 41 = 178 possibilities.

b) How many distinct ways are there of choosing one discrete structures student from each class to form an advisory committee?

Use the product rule. 73*64*41 = 191552

c) How many distinct ways are there of listing six different discrete structures students to form the 41 person section to go to the board one after the other to present problem solutions?

Rosen's version of the product rule applies most directly here. There are 41 students to choose from as the first presenter but then there are only 40 students to choose from as the second presenter, 39 as the third presenter and so on. The result is

41*40*39*38*37*36 = 3237399360

There is more discussion below of this kind of problem in the "permutations" section.

d) How many distinct ways are there of choosing six discrete structures students to form the course volleyball team?

Now we have to choose a set of six students out of the 178 discrete structures students. If we count the ways to make a list of 6 students, as in c, we get

178*177*176*175*174*173 possible ordered list of 6 students.

Each set of 6 students appears in this list 6! = 6*5*4*3*2*1 times so the number of sets is of six students out of the 178 is

$$\frac{178*177*176*175*174*173}{6*5*4*3*2*1} = \frac{178!}{(172!)(6!)}$$

There is more discussion below of this kind of problem in the "combinations" section.

More Passwords:

Suppose passwords are restricted to 6 case-sensitive alpha-numeric characters and must contain at least 1 digit and at least 1 letter. How many are there?

There are $(62)^6$ passwords composed of 6 case-sensitive alpha-numeric characters with no other restrictions. Of these, $(52)^6$ are composed of letters only and $(10)^6$ are composed of digits only. All the others have at least one digit and at least one letter. So the answer is $[(62)^6 - (52)^6 - (10)^6]$.

Suppose passwords may have 6 to 10 case-sensitive alpha-numeric characters and must contain at least 1 digit and at least 1 letter. How many are there?

Since a password may have 6 or 7 or 8 or 9 or 10 letters, we can count each of these possibilities separately and apply the sum rule to get the result.

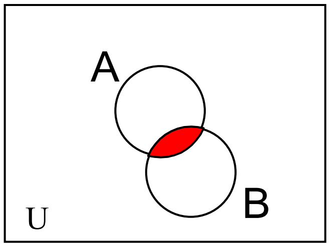
$[(62)^6 - (52)^6 - (10)^6] + [(62)^7 - (52)^7 - (10)^7] + [(62)^8 - (52)^8 - (10)^8] + [(62)^9 - (52)^9 - (10)^9] + [(62)^{10} - (52)^{10} - (10)^{10}]$

NSF Fast-Lane Passwords have 6 to 10 alpha-numeric characters where upper- and lower-case are distinguished. They must contain at least 2 digits and at least 2 letters. How many are there?

Inclusion-Exclusion Principle:

If A and A are finite sets then $|A \cup B| = |A| + |B| - |A \cap B|$.

|A| + |B| counts the elements in A and the elements in B but elements of $A \cap B$ (the red ones), are counted twice.



Examples:
How many strings of 6 upper-case letters start with A or end with Z?
26^5 start with A. 26^5 end with Z. 26^4 start with A and end with Z so they were counted twice. The answer is $26^5 + 26^5 - 26^4$.

This problem is from Rosen, Exercise 20, page 311.
How many positive integers between 1000 and 9999 inclusive
(There are 9000 consecutive integers.)
a) are divisible by 9?
b) are even?
c) have distinct digits?
d) are not divisible by 3?
e) are divisible by 5 or 7? Use the Inclusion-Exclusion Principle here.
f) are not divisible by either 5 or 7?
g) are divisible by 5 but not by 7?
h) are divisible by 5 and 7?

Pigeonhole Principle

If k + 1 or more objects are placed in k boxes, then there is one box that has two or more objects.

Examples:

a) There are 102 students in my two sections of CSU200. If they all take the final, at least two of them will get the same grade?

There are 101 possible grades 0, 1, . . ., 100 so the result follows from the pigeonhole principle.

b) If I use the last two digits of their social security numbers as a code to post grades in anonymity, at least two students will get the same code?

There are 100 2-digit codes, 00 through 99 so by the time I list the first 101 students there will be two with the same grade.

c) If a drawer contains 12 red socks and 12 blue socks and I pull some socks out in the dark, how many must I pull out to be sure of having a pair?

d) Every integer *n* has a multiple that has only 0s and 1s in its decimal expansion.
Proof
Consider the n+1 numbers 1, 11, ... , 11111...11 where the last number has (n-1) ones. If we evaluate each of these numbers mod n, two of them must give the same value as there are only n possible results, 0, ..., n-1. If $a \bmod n = b \bmod n$ then *a* - *b* is divisible by n. So take the two numbers that result in the same value and subtract the smaller from the larger. The result is a multiple of *n* and has only 0s and 1s in its decimal expansion.

Here's an example. Take n=6. $\begin{array}{l} 1 \bmod 6 = 1 \\ 11 \bmod 6 = 5 \\ 111 \bmod 7 = 3 \\ 1111 \bmod 7 = 1 \end{array}$ so 1111=1 = 1110 is a multiple of 6.

e) In any set of n +1 positive integers not exceeding 2n, there must be one integer that divides another.
Proof:
Write each integer as a power of 2 times an odd integer, $a_j = 2^{e_j} q_j$. Then $q_1 \ldots q_{n+1}$ are n+1 odd integers < 2n. Two of them must be the same. One of the corresponding a_js divides the other.

Generalized Pigeonhole Principle

If N objects are placed in k boxes, at least one box contains $\lceil N/k \rceil$ objects.
Proof: $k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N.$

a) There are 51 students in this class. At least ____ students in this class have birthdays in the same month. What is the largest number I can put in the blank and be sure the statement is true.

$\lceil 64/12 \rceil = 5$ have birthdays in the same month.

Permutations

A **permutation** of a set of objects is an ordered arrangement of those objects.
An **r-Permutation** is and ordered arrangement of r elements of a set.

Theorem: The number of r-permutations of a set with n distinct elements is

$$\textbf{P(n,r) = n(n-1)(n-2) . . . (n – r + 1)} = \frac{n!}{(n-r)!}.$$

Proof:

1	n choices for first place
2	n-1 choices left for second place
...	
r-1	n-(r-2) choices left for (r-1)th place
r	n-(r-1) choices left for rth place

Now use the product rule.

Examples:

a) A wedding party consists of the bride, the groom, the bride's mother and father, the groom's mother and father, the best man, the maid of honor, two ushers, and two bride's maids.

i) How many ways are there of arranging all of them in a row for a picture?

There are 12 people in the wedding party so there are P(12,12) = 12! ways of arranging them in a row.

ii) How many ways if the bride and groom stand together on the left side of the line?

There are 2 ways to arrange the bride and groom on the far left side of the line and P(10,10) = 10! ways of arranging the rest of the party so 2*10! possible arrangements.

iii) How many ways if the bride and groom are together but anywhere in the line?

There are P(10,10) = 10! ways of arranging the rest of the party without the bride and groom. Then the bride and groom together can be place between and two of the lined up people or to the left or to the right of all of them. That's 11 different positions. There are 2 ways to arrange the bride and groom. THe total number of arrangements is (10!)*11*2.

iv) How many ways can 5 members of the wedding party line up for a picture?

We must line up 5 people out of 12 so P(12, 5) = 12*11*10*9*8.

b) On the trip I am about to take, I must visit Florence, Milan, Venice, London, Bristol, and Warwick.

i) How many itineraries are possible?

There are 6 cities so 6! possible itineraries.

ii) How many itineraries are possible if all the British cities are consecutive and all the Italian cities are consecutive?

Florence, Milan, and Venice are in Italy. London, Bristol, and Warwick are in England. There are 3! orders for the Italian cities and 3! orders for the British cities. I can go to Italy first or to England first so there are (3!)*(3!)*2 = 72 possible itineraries.

Strings

a) How many permutation are there of the letters A B C D E F G H I J?

P(10,10) = 10!

b) How many of them contain the block

i) HEAD?

First arrange the other letters, B, C, F, G, I, J. There are 6! arrangements.
Then place the block HEAD between two of the arranged letters or at one of the ends. There are 7 places it can go. That makes a total of 7*6!.

ii) HJF?

Just like part i but you must arrange the other 7 letters and then there are 8 places where HJF can go.

iii) BIGFACEDHJ?

Just one.

Combinations

An **r-combination** of elements of a set is an unordered selection of r elements of the set, i.e. a subset of the set with r elements.

$C(n,r)=\binom{n}{r}$ is the number of r-combinations of a set with n elements.

Theorem: If $0 \le r \le n$ are integers, $C(n,r)=\binom{n}{r}=\frac{n!}{r!(n-r)!}$.

Proof:
P(n, r) = C(n, r)P(r, r). That is, to create an ordered list of r elements from a set of n elements, first choose r elements from the set (There are C(n, r) ways to do this.) and then choose an ordering of the r elements (There are P(r, r) ways to do this.). Since we already know that $P(n,r)=\frac{n!}{(n-r)!}$ and $P(r,r)=\frac{r!}{(0)!}=r!$, the result follows.

Corollary: If 0 ≤ r ≤ n are integers, then C(n, r) = C(n, n-r).

Examples:

a) Eight members of the wedding party (described above) are to do a traditional circle dance. How many different groups of eight can be selected?

There are 12 people in the wedding party so we can choose C(12, 8) different subsets of 8 people. $C(12,8)=\frac{12!}{(8!)(4)!}=\frac{12\cdot 11\cdot 10\cdot 9}{4\cdot 3\cdot 2\cdot 1}=11\cdot 5\cdot 9=495.$

b) Now that we have selected 8 people for the dance, how many ways can we arrange them in a circle?

This is a permutation problem, not a combination problem. It is similar, but not quite the same as finding the number of ways to arrange 8 people in a line. There are P(8, 8) = 8! ways to do that. Each circular arrangement will appear 8 times as a linear arrangement. (A B C D E F G H forms the same circular arrangement as B C D E F G H A or C D E F G H A B, . . .) So there are 8!/8 = 7! possible circle dance arrangements.

c) How may ways can I select 3 men and 3 women from the wedding party?

There are 6 men and 6 women in the wedding party. The number of ways of choosing 3 men (or 3 women) is C(6, 3) = 20. The number of ways of selecting 3 men and 3 women from the wedding party is 20*20 = 400.

d) How many ways can I select 6 students from this class of 51 students to get a grade of "A"?

$$C(51,6) = \frac{51!}{(45!)(6)!} = \frac{51\cdot 50\cdot 49\cdot 48\cdot 47\cdot 46}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 17\cdot 10\cdot 49\cdot 2\cdot 47\cdot 46 = 36018920.$$

Really, I won't do it this way; I **will** look at you grades.

e) How many bytes contain exactly three 1's?

A byte is an 8-bit string so there are $2^8 = 256$ bytes total. Picking a byte with exactly three 1's is the same as selecting a subset of size 3 out of the 8 bit positions. The number of ways of doing this is

$$C(8,3) = \frac{8!}{(5!)(3)!} = \frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1} = 8\cdot 7 = 56.$$

Binomial Theorem

Let x and y be variables and n a positive integer. Then

$$(x+y)^n = \sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

Proof:

Count the number of times $x^{n-j}y^j$ appears in the product.

$$\underbrace{(x+y)(x+y)(x+y)\cdots(x+y)(x+y)}_{\text{n copies}}.$$

Each appearance of $x^{n-j}y^j$ is this product corresponds to a *j*-size subset of the *n* positions. We take *y* from each of those *j* positions and *x* from the remaining *n-j* positions. The product $x^{n-j}y^j$ will appear $C(n,j) = \binom{n}{j}$ times.

Examples:

a) Expand $(x+y)^4$.

$$(x+y)^4 = \sum_{j=0}^{4}\binom{4}{j}x^{4-j}y^j = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

b) Expand $(2x + y^{-2})^3$.

$$(2x+y^{-2})^3 = \sum_{j=0}^{3}\binom{3}{j}(2x)^{3-j}(y^{-2})^j$$

$$= \binom{3}{0}(2x)^3 + \binom{3}{1}(2x)^2(y^{-2}) + \binom{3}{2}(2x)(y^{-2})^2 + \binom{3}{3}(y^{-2})^3$$

$$= (2x)^3 + 3(2x)^2(y^{-2}) + 3(2x)(y^{-2})^2 + (y^{-2})^3$$

$$= 8x^3 + 12x^2y^{-2} + 6xy^{-4} + y^{-6}.$$

c) Give the term in $(a+b)^{42}$ that has *b* to the 17 power. $\binom{42}{17}a^{25}b^{17}$

Theorem: If n is a positive integer and $0 \le k < n$, then $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof:

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{(n-k)!k!} + \frac{n!}{(n-(k-1))!(k-1)!}$$

$$= \frac{(n-(k-1))n!}{(n-(k-1))(n-k)!k!} + \frac{k(n!)}{k(n-(k-1))!(k-1)!}$$

$$= \frac{(n-(k-1)+k)n!}{(n-(k-1))!k!} = \frac{(n+1)n!}{(n+1-k)!k!} = \frac{(n+1)!}{(n+1-k)!k!} = \binom{n+1}{k}.$$

This theorem tells you that you can compute an entry in Pascal's triangle by adding the two elements diagonally above it.

$(a+b)^0$ **1**

$(a+b)^1$ **1 1**

$(a+b)^2$ **1 2 1**

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$(a+b)^3$ **1 3 3 1**

$(a+b)^4$ **1 4 6 4 1**

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$(a+b)^5$ **1 5 10 10 5 1**

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$(a+b)^6$ **1 6 15 20 15 6 1**

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$(a+b)^7$ **1 7 21 35 35 21 7 1**

References:

Paul A. Foerster, "Algebra and Trigonometry: Functions and Applications," Addison-Wesley, 1980.

James L. Hein, "Discrete Mathematics, second edition," Jones and Bartlett Mathematics, 2003.

Kenneth H. Rosen, "Discrete Mathematics and its Applications, 5^{th} edition," McGraw-Hill, 2003.