Solutions for Written Homework 4

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1. (12 points) The Successor of a Set
The *successor* of a set S is the set S $\cup$ {S}.
a) Give the successor of each of these sets.

- **i)** S = {1} therefore: successor(S) = {1, {1}}.
- **ii)** S = {1, 2} therefore: successor(S) = {1, 2, {1,2}}.
- **iii)** S = Ø therefore: successor(S) = { Ø }.
- **iv)** S = {Ø} therefore: successor(S) = { Ø, { Ø }}.
- **v)** S = {Ø, {Ø}} therefore: successor = {Ø, {Ø}, {Ø, {Ø}}}

b) If the set S has *n* elements, how many elements does the successor of S have? Explain your answer.

successor(S) has n+1 elements. A set is one element. We can have a set P of sets. The size of set P is the number of elements in P. Each element in P is a set. For example,
P = {{1, 2}, {3, 4, 5}} has 2 elements, not 5.

2. (16 points) Relations
For each of the following, give a relation Q on the real numbers satisfying the given conditions or explain why no such relation exists. (Explanations in answers are examples of when a relation property fails)

a. **no, no, no:** y = √x
Note that: 2 ≠ √2; 2 = √4 does not imply 4 = √2; 2 = √4 and 4 = √16 do not imply that 2 = √16
b. **no, no, yes:** x < y
Note that: x<x is false; 1<2 but 2<1 is false
c. **no, yes, no:** x ≠ y
Note that: x ≠ x is false; x ≠ y and y ≠ z says that x ≠ z, but they could be equal.
d. **no, yes, yes:** x > 0 and y > 0
Note that: (0,0) is not in the set;
e. **yes, no, no:** x = y with points (3,0) and (1,0)
Note that: (3,0) is in the set but (0,3) is not; (3,0) and (0,1) imply (3,1), but (3,1) is not in the set.
f. **yes, no, yes:** x ≤ y
Note that: 3 ≤ 4 but 4 ≤ 3 is false.
g. **yes, yes, no:** x=y, y = x+1, y = x-1
Note that: (0,1) and (1,2) imply (0,2) but (0,2) is not in this set.
h. **yes, yes, yes:** x = y

3. (8 points) Expression Trees Given the expression: $((x+y)/2)*(3-(w-a)*(5+c))$,
a) Draw the corresponding expression tree.

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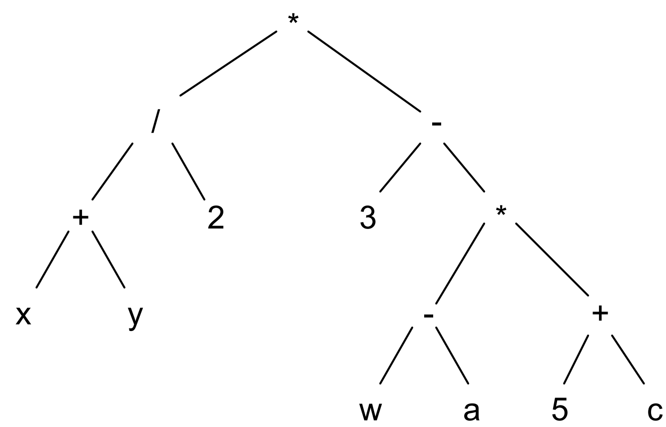

b) Give the Scheme expression that corresponds to the tree.

(* (/ (+ x y) 2) (- 3 (* (- w a) (+ 5 c))))

4. (4 points) (Size of Binary Trees) Prove by Induction:
Theorem: If T is a binary tree of height *h,* then T has at most 2^{h+1}-1 nodes.

The binary tree with height *h* can be full or it can be an unbalanced chain.
We want to show that if T is a binary tree of height *h,* then T has at most 2^{h+1}-1 nodes.

Base Case: When the height of T is 0, then all we have is a root which is one node. We see that $2^{0+1}-1 = 2-1 = 1$. Next try $h = 1$. We have a tree with at most 3 nodes, a root and two children nodes. We see that $2^{i+1}-1 = 4-1 = 3$.

Induction Step: Assume that the theorem is true up to $h = i$. We want to show that a tree with height $h = i+1$ has at most $2^{(i+1)+1}-1 = 2^{i+2}-1$.

Suppose we have a tree of height $h=i+1$. Then both the left and right subtrees have height $h=i$ and have at most 2^{i+1}-1 nodes each. If we add the nodes of both subtrees and the root we get:

size(i+1) = root + size(left subtree) + size(right subtree)
$= 1 + 2^{i+1} - 1 + 2^{i+1} - 1 = 2*2^{i+1} - 1= 2^{i+1+1} - 1=2^{(i+1)+1}-1 = 2^{i+2} - 1.$

So we have that a tree of height $h = i+1$ has at most $2^{(i+1)+1}-1$.
Therefore, If T is a binary tree of height *h,* then T has at most 2^{h+1}-1 nodes.