1. **Set-Builder Notation**

Rewrite each set showing all its elements, for example, \( \{ x \in \mathbb{Z} \mid 1 < x < 5 \} = \{2, 3, 4\} \).

a) \( \{ x \in \mathbb{Z} \mid x \mod 4 = 3 \text{ and } |x| \leq 12 \} \)
\{-9, -5, -1, 3, 7, 11\}

b) \( \{ x \in \mathbb{Z} \mid 0 \leq x/4 \leq 2 \} \)
\{0, 1, 2, 3, 4, 5, 6, 7, 8\}

2. **Cartesian Product and Power Sets**

Let \( A = \{1, 2\} \) and \( B = \{1, 2, 3\} \).

a) List all the elements of \( A \times B \).
(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)

b) List all the members of \( \mathcal{P}(B) \), the power set of \( B \).
\( \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \)

c) List all the subsets of \( A \times A \) that have two elements.
The elements of \( A \times A \) are \((1, 1), (1, 2), (2, 1), (2, 2)\).
The subsets of \( A \times A \) that have two elements are
\( \{(1, 1), (1, 2)\}, \{(1, 1), (2, 1)\}, \{(1, 1), (2, 2)\}, \{(1, 2), (2, 1)\}, \{(1, 2), (2, 2)\}, \{(2, 1), (2, 2)\} \)

d) How many subsets does \( \mathcal{P}(A) \) have?
\( \mathcal{P}(A) \) has 4 elements \( \emptyset, \{1\}, \{2\}, \{1, 2\} \).
\( \mathcal{P}(A) \) has \( 2^4 = 16 \) subsets.

e) What is the cardinality of \( B \times \mathcal{P}(A) \)?
\( |B \times \mathcal{P}(A)| = |B| \times |
\mathcal{P}(A)| = 3 \times 4 = 12 \).

3. **Summation Notation**

a) Expand each of the following sums as shown in the example.

\[
\sum_{k=1}^{4} 2k^2 = 2(1)^2 + 2(2)^2 + 2(3)^2 + 2(4)^2 = 2 + 8 + 18 + 32
\]

i) \[
\sum_{k=1}^{3} 5^k = 5^1 + 5^2 + 5^3 = 5 + 25 + 125
\]

ii) \[
\sum_{k=1}^{5} \frac{k!}{3!} = \frac{1!}{3!} + \frac{2!}{3!} + \frac{3!}{3!} + \frac{4!}{3!} + \frac{5!}{3!} = \frac{1}{6} + \frac{1}{3} + 4 + 20
\]

iii) \[
\sum_{k=3}^{5} (k^2 - (k-1)^2) = \left(3^2 - (3-1)^2\right) + \left(4^2 - (4-1)^2\right) + \left(5^2 - (6-1)^2\right) = 5 + 7 + 9
\]
4. **Summation Formulas**
For each of the following sums, give a formula in terms of \( n \) for the sum, as shown in the example.

\[
\sum_{k=1}^{n} 3k = \frac{3n(n + 1)}{2}
\]

i) \[
\sum_{k=1}^{n} 3^k = \frac{3^{n+1} - 3}{2}
\]

Set \( S = \sum_{k=1}^{n} 3^k \). Then \( 3S = \sum_{k=2}^{n+1} 3^k \). Therefore, \( 2S = \sum_{k=2}^{n+1} 3^k - \sum_{k=1}^{n} 3^k = 3^{n+1} - 3 \)

ii) \[
\sum_{k=10}^{n} 5k = \]

where \( n \geq 10 \)

This is arithmetic. Use the formula \( \frac{(first + last) \times (# terms)}{2} \)

5. **Permutations and Combinations**
Show the formulas you used as well as the final numbers.

Give the values of each of these quantities:

- a) \( P(6, 2) = \frac{6!}{4!} = 6 \cdot 5 = 30 \)
- b) \( P(7, 4) = \frac{7!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840 \)
- c) \( C(6, 2) = \frac{6!}{4!2!} = \frac{6 \cdot 5}{2} = 15 \)
- d) \( C(7, 4) = \frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 35 \)
- e) \( C(9, 9) = \frac{9!}{0!9!} = \frac{9!}{1 \cdot 9!} = 1 \)
6. **Counting**

Show the formulas you used as well as the final numbers.

If you have 8 books, how many ways can you

a) arrange 3 of them on a shelf?

\[ P(8,3) = \frac{8!}{5!} = 8 \cdot 7 \cdot 6 = 336 \]

b) choose 4 of them to take on a trip?

\[ C(8,4) = \frac{8!}{4!4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2} = 2 \cdot 7 \cdot 5 = 70 \]

c) choose 3 of them to leave home?

\[ C(8,3) = \frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} = 8 \cdot 7 = 56 \]

d) choose 1 a day to pick a quote from for the next 3 days, repetition allowed?

You have 8 choices on each day so \( 8^3 = 512 \).

7. **Probability**

Show your work as well as the final numbers.

If you roll a pair of dice, one black and one white, what is the probability of

a) the sum of the numbers rolled is 4?

**successful outcomes**

<table>
<thead>
<tr>
<th>Black die</th>
<th>White die</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

There are 3 successful outcomes in all. There are 6×6=36 possible outcomes so the probability is \( 3/36 = 1/12 \).

b) the sum of the numbers rolled is 11 or 12?

**successful outcomes**

<table>
<thead>
<tr>
<th>Black die</th>
<th>White die</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

There are 3 successful outcomes in all. There are 6×6=36 possible outcomes so the probability is \( 3/36 = 1/12 \).

b) the number on the white one is greater than the number on the black one?

**successful outcomes**

<table>
<thead>
<tr>
<th>White die</th>
<th>Black die</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>5, 4, 3, 2, 1</td>
</tr>
<tr>
<td>5</td>
<td>4, 3, 2, 1</td>
</tr>
<tr>
<td>4</td>
<td>3, 2, 1</td>
</tr>
<tr>
<td>3</td>
<td>2, 1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>no possibilities</td>
</tr>
</tbody>
</table>

There are 15 successful outcomes in all. There are 6×6=36 possible outcomes so the probability is \( 15/36 = 5/12 \).
8. Relations
Let A be the set \{-3, -2, -1, 0, 1, 2, 3\}.

a) Show the pairs \((x, y)\) in the relation \(|y| = |x|\) by putting an \(\times\) in the corresponding squares:

\[
\begin{array}{cccccc}
-3 & -2 & -1 & 0 & 1 & 2 \\
\times & \times & \times & \times & \times & \times \\
-2 & -1 & 0 & 1 & 2 & 3 \\
\times & \times & \times & \times & \times & \times \\
-1 & 0 & 1 & 2 & 3 & 4 \\
\times & \times & \times & \times & \times & \times \\
0 & 1 & 2 & 3 & 4 & 5 \\
\times & \times & \times & \times & \times & \times \\
1 & 2 & 3 & 4 & 5 & 6 \\
\times & \times & \times & \times & \times & \times \\
2 & 3 & 4 & 5 & 6 & 7 \\
\times & \times & \times & \times & \times & \times \\
3 & 4 & 5 & 6 & 7 & 8 \\
\times & \times & \times & \times & \times & \times \\
\end{array}
\]

b) Tell whether this relation is
i) Reflexive - Yes, \(|x| = |x|\). You could also note that the line \(y = x\) is part of the relation.

ii) Symmetric - Yes if \(|x| = |y|\) then \(|y| = |x|\).

ii) Transitive - Yes, if \(|x| = |y|\) and \(|y| = |z|\) then \(|x| = |z|\).

ii) Symmetric

ii) Transitive

Explain your answers.