SOLUTION

College of Computer and Information Science **CS U200 Discrete Structures**
Professor Harriet Fell **Third Hour Exam –Section 2** **November 22, 2004**

1. Set-Builder Notation

Rewrite each set showing all its elements, for example, $\{ x \in Z \mid 1 < x < 5\} = \{2, 3, 4\}$.

a) $\{ x \in Z \mid x \textbf{ mod } 4 = 3 \text{ and } |x| <= 12 \}$
$\{-9, -5, -1, 3, 7, 11\}$

b) $\{ x \in Z \mid 0 <= x/4 <= 2 \}$
$\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

2. Cartesian Product and Power Sets

Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$.

a) List all the elements of $A \times B$.
(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)

b) List all the members of $\mathcal{P}(B)$, the power set of B.
$\varnothing$, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}

c) List all the subsets of $A \times A$ that have two elements.
The elements of $A \times A$ are (1, 1), (1, 2), (2, 1), (2, 2).
The subsets of $A \times A$ that have two elements are
{(1, 1), (1, 2)}, {(1, 1), (2, 1)}, {(1, 1), (2, 2)},
{(1, 2), (2, 1)}, {(1, 2), (2, 2)}, {(2, 1), (2, 2)}

d) How many subsets does $\mathcal{P}(A)$ have?
$\mathcal{P}(A)$ has 4 elements $\varnothing$, {1}, {2}, and {1, 2}.
$\mathcal{P}(A)$ has 2^4 subsets.

e) What is the cardinality of $B \times \mathcal{P}(A)$?
$|B \times \mathcal{P}(A)| = |B| \times |\mathcal{P}(A)| = 3 \times 4 = 12$.

3. Summation Notation

a) Expand each of the following sums **as shown in the example**.

$$\sum_{k=1}^{4} 2k^2 = 2(1)^2 + 2(2)^2 + 2(3)^2 + 2(4)^2 = 2 + 8 + 18 + 32$$

i) $$\sum_{k=1}^{3} 5^k = 5^1 + 5^2 + 5^3 = 5 + 25 + 125$$

ii) $$\sum_{k=1}^{5} \frac{k!}{3!} = \frac{1!}{3!} + \frac{2!}{3!} + \frac{3!}{3!} + \frac{4!}{3!} + \frac{5!}{3!} = \frac{1}{6} + \frac{1}{3} + 1 + 4 + 20$$

iii) $$\sum_{k=3}^{5} \left(k^2 - (k-1)^2\right) = \left(3^2 - (3-1)^2\right) + \left(4^2 - (4-1)^2\right) + \left(5^2 - (6-1)^2\right) = 5 + 7 + 9$$

4. Summation Formulas

For each of the following sums, give a formula in terms of ***n*** for the sum, as shown in the example.

$$\sum_{k=1}^{n} 3k = \frac{3n(n+1)}{2}$$

i) $\sum_{k=1}^{n} 3^k = \frac{3^{n+1}-3}{2}$

Set $S = \sum_{k=1}^{n} 3^k$. Then $3S = \sum_{k=2}^{n+1} 3^k$. Therefore, $2S = \sum_{k=2}^{n+1} 3^k - \sum_{k=1}^{n} 3^k = 3^{n+1} - 3$

ii) $\sum_{k=10}^{n} 5k =$

where $n \geq 10$

This is arithmetic. Use the formula $\frac{(first + last) \times (\#terms)}{2}$

5. Permutations and Combinations

Show the formulas you used as well as the final numbers.

Give the values of each of these quantities:

a) $P(6, 2) = \frac{6!}{4!} = 6 \cdot 5 = 30$

b) $P(7, 4) = \frac{7!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$

c) $C(6, 2) = \frac{6!}{4!2!} = \frac{6 \cdot 5}{2} = 15$

d) $C(7, 4) = \frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 35$

e) $C(9, 9) = \frac{9!}{0!9!} = \frac{9!}{1 \cdot 9!} = 1$

6. Counting

Show the formulas you used as well as the final numbers.

If you have 8 books, how many ways can you

a) arrange 3 of then on a shelf?

$$P(8,3) = \frac{8!}{5!} = 8 \cdot 7 \cdot 6 = 336$$

b) choose 4 of them to take on a trip?

$$C(8,4) = \frac{8!}{4!4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2} = 2 \cdot 7 \cdot 5 = 70$$

c) choose 3 of them to leave home?

$$C(8,3) = \frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} = 8 \cdot 7 = 56$$

d) choose 1 a day to pick a quote from for the next 3 days, repetition allowed?

You have 8 choices on each day so $8^3 = 512$.

7. Probability

Show your work as well as the final numbers.

If you roll a pair of dice, one black and one white, what is the probability of

a) the sum of the numbers rolled is 4?

successful outcomes

Black die	White die
1	3
2	2
3	1

There are 3 successful outcomes in all. There are 6×6=36 possible outcomes so the probability is 3/36 = 1/12.

b) the sum of the numbers rolled is 11 or 12?

successful outcomes

Black die	White die
5	6
6	5
6	6

There are 3 successful outcomes in all. There are 6×6=36 possible outcomes so the probability is 3/36 = 1/12.

c) the number on the white one is greater than the number on the black one?

successful outcomes

White die	Black die
6	5, 4, 3, 2, 1
5	4, 3, 2, 1
4	3, 2, 1
3	2, 1
2	1
1	no possibilities

There are 15 successful outcomes in all. There are 6×6=36 possible outcomes so the probability is 15/36 = 5/12.

8. Relations

Let A be the set {-3, -2, -1, 0, 1, 2, 3}.

a) Show the pairs (x, y) in the relation $|y| = |x|$ by putting an × in the corresponding squares:

3	×						×
2		×				×	
1			×		×		
0				×			
-1			×		×		
-2		×				×	
-3	×						×
	-3	-2	-1	0	1	2	3

b) Tell whether this relation is

i) Reflexive - Yes, $|x| = |x|$. You could also note that the line y = x is part of the relation.

ii) Symmetric - Yes if $|x| = |y|$ then $|y| = |x|$.

ii) Transitive - Yes, if $|x| = |y|$ and $|y| = |z|$ then $|x| = |z|$.

ii) Symmetric

ii) Transitive

Explain your answers.