SOLUTION

College of Computer and Information Science **CS U200 Discrete Structures**
Professor Harriet Fell **Third Hour Exam –Section 1** **November 22, 2004**

1. Set-Builder Notation

Rewrite each set showing all its elements, for example, $\{ x \in Z \mid 1 < x < 5\} = \{2, 3, 4\}$.

a) $\{ x \in Z \mid x \textbf{ mod } 3 = 2 \text{ and } |x| <= 10 \}$
$\{-10, -7, -4, -1, 2, 5, 8\}$

b) $\{ x \in Z \mid 0 <= x/3 <= 2 \}$
$\{0< 1, 2, 3, 4, 5, 6\}$

2. Cartesian Product and Power Sets

Let $A = \{1, 2\}$ and $B = \{2, 3, 4\}$.

a) List all the elements of $A \times B$.
$(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)$

b) List all the members of $\mathcal{P}(A)$, the power set of A.
$\varnothing, \{1\}, \{2\}, \{1, 2\}$

c) List all the subsets of $A \times A$ that have two elements.
The elements of $A \times A$ are $(1, 1), (1, 2), (2, 1), (2, 2)$.
The subsets of $A \times A$ with two elements are
$\{(1, 1), (1, 2)\}, \{(1, 1), (2, 1)\}, \{(1, 1), (2, 2)\},$
$\{(1, 2), (2, 1)\}, \{(1, 2), (2, 2)\}, \{(2, 1), (2, 2)\}.$

d) How many subsets does $\mathcal{P}(B)$ have?
$\mathcal{P}(B)$ has 8 elements so $\mathcal{P}(B)$ has $2^8 = 256$ subsets.

e) What is the cardinality of $A \times \mathcal{P}(B)$?
$|A \times \mathcal{P}(B)| = |A| \times |\mathcal{P}(B)| = 2 \times 8 = 16.$

3. Summation Notation

a) Expand each of the following sums **as shown in the example**.

$$\sum_{k=1}^{4} 2k^2 = 2(1)^2 + 2(2)^2 + 2(3)^2 + 2(4)^2 = 2 + 8 + 18 + 32$$

i) $$\sum_{k=1}^{4} 2^k = 2^1 + 2^2 + 2^3 + 2^4 = 2 + 4 + 8 + 16$$

ii) $$\sum_{k=1}^{5} \frac{k!}{2!} = \frac{1!}{2!} + \frac{2!}{2!} + \frac{3!}{2!} + \frac{4!}{2!} + \frac{5!}{2!} = \frac{1}{2} + 1 + 3 + 12 + 60$$

iii) $$\sum_{k=3}^{5} \left(k^2 - k + 1\right) = \left(3^2 - 3 + 1\right) + \left(4^2 - 4 + 1\right) + \left(5^2 - 5 + 1\right) = 7 + 13 + 21$$

4. Summation Formulas

For each of the following sums, give a formula in terms of ***n*** for the sum, as shown in the example.

$$\sum_{k=1}^{n} 3k = \frac{3n(n+1)}{2}$$

i) $$\sum_{k=1}^{n} 3^k = \frac{3^{n+1}-3}{2}$$

Set $S = \sum_{k=1}^{n} 3^k$. Then $3S = \sum_{k=2}^{n+1} 3^k$. Therefore, $2S = \sum_{k=2}^{n+1} 3^k - \sum_{k=1}^{n} 3^k = 3^{n+1} - 3$

ii) $$\sum_{k=10}^{n} 5k = \frac{(50+5n)(n-9)}{2}$$

where $n \geq 10$

This is arithmetic. Use the formula $\frac{(first + last) \times (\#terms)}{2}$

5. Permutations and Combinations

Show the formulas you used as well as the final numbers.

Give the values of each of these quantities:

a) $P(8, 2) = \frac{8!}{6!} = 8 \cdot 7 = 56$

b) $P(7, 3) = \frac{7!}{4!} = 7 \cdot 6 \cdot 5 = 210$

c) $C(8, 2) = \frac{8!}{6!2!} = \frac{8 \cdot 7}{2} = 28$

d) $C(7, 3) = \frac{7!}{4!3!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 35$

e) $C(9, 1) = \frac{9!}{8!1!} = 9$

6. Counting

Show the formulas you used as well as the final numbers.

If you have 6 books, how many ways can you

a) arrange 4 of then on a shelf?

$$P(6,4) = \frac{6!}{2!} = 6 \cdot 5 \cdot 4 \cdot 3 = 360$$

b) choose 4 of them to take on a trip?

$$C(6,4) = \frac{6!}{2!4!} = \frac{6 \cdot 5}{2} = 15$$

c) choose 3 of them to leave home?

$$C(6,3) = \frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2} = 20$$

d) choose 1 a day to pick a quote from for the next 3 days, repetition allowed?

You have 6 choices on each day so $6^3 = 216$.

7. Probability

Show your work as well as the final numbers.

If you roll a pair of dice, one black and one white, what is the probability of

a) the sum of the numbers rolled is 8?

successful outcomes

Black die	White die
2	6
3	5
4	4
5	3
6	2

There are 5 successful outcomes in all. There are 6×6=36 possible outcomes so the probability is 5/36.

b) the sum of the numbers rolled is 2 or 3?

successful outcomes

Black die	White die
1	1
1	2
2	1

There are 3 successful outcomes in all. There are 6×6=36 possible outcomes so the probability is 3/36 = 1/12.

c) the number on the white one is less than the number on the black one?

successful outcomes

Black die	White die
6	5, 4, 3, 2, 1
5	4, 3, 2, 1
4	3, 2, 1
3	2, 1
2	1
1	no possibilities

There are 15 successful outcomes in all. There are 6×6=36 possible outcomes so the probability is 15/36 = 5/12.

8. Relations

Let A be the set {-3, -2, -1, 0, 1, 2, 3}.

a) Show the pairs (x, y) in the relation $x^2 = y^2$ by putting an × in the corresponding squares:

	-3	-2	-1	0	1	2	3
3	×						×
2		×				×	
1			×		×		
0				×			
-1			×		×		
-2		×				×	
-3	×						×

b) Tell whether this relation is

i) Reflexive - Yes, $x^2 = x^2$. You could also note that the line y = x is part of the relation.

ii) Symmetric - Yes if $x^2 = y^2$ then $y^2 = x^2$.

ii) Transitive - Yes, if $x^2 = y^2$ and $y^2 = z^2$ then $x^2 = z^2$.

Explain your answers.