SOLUTION

College of Computer and Information Science **CS U200 Discrete Structures**
Professor Harriet Fell **First Hour Exam –Section 2** **September 29, 2004**

1. (10 points) For each of the following, give a real number that satisfies the given conditions.

a. A number that is in $\mathbb{Z}$ but not in $\mathbb{R}^+$. -3

b. A number that is in $\mathbb{R}^+$ but not in $\mathbb{Z}$. 1/3

c. A number that is in $\mathbb{R}^-$ but not in $\mathbb{Q}$. $-\sqrt{3}$

d. A number that is in $\mathbb{Q}$ but not in $\mathbb{Z}$. 1/3

e. A number that is in $\mathbb{Z}$ but not in $\mathbb{N}$. 0 or any negative integer

2. (8 points) Which of the following sets of numbers are fields? For those that are not fields, tell all the field axioms that do not hold.

a. $\mathbb{R}^+$

Not a field. $\mathbb{R}^+$ does not contain an additive identity (0 is not in $\mathbb{R}^+$**)** and does not contain additive inverses (-3 is not in $\mathbb{R}^+$**)**.

$\mathbb{R}^+$ is closed under + and ×. The commutative, associative, and distributive laws hold as this is just regular real arithmetic.

$\mathbb{R}^+$ has a multiplicative identity 1 and every element in $\mathbb{R}^+$ has a multiplicative inverse in $\mathbb{R}^+$.

b. The set of all rationals $\frac{a}{b}$ where a and b are integers and a is a power of 2.

This set is not closed under addition, for example,

$$\frac{2}{3}+\frac{1}{5}=\frac{13}{15}$$

Though 13/15 can be expressed as a ratio of integers in many ways, e.g. 26/30, the numerator will always have 13 as a divisor.

The set is closed under multiplication $\frac{2^m}{a}\bullet\frac{2^n}{b}=\frac{2^{m+n}}{ab}$.

The commutative, associative, and distributive laws hold as this is just regular real arithmetic

0 is not a power of 2 so this set does not contain an additive identity.

1 is a power of 2 so this set does contain a multiplicative identity.

Every element of this set $\frac{2^m}{a}$ has an additive inverse in the set $\frac{2^m}{-a}$.

Some elements of this set do not have multiplicative inverses in this set, e.g. the multiplicative inverse of 2/3 is 3/2 or 3*a*/2*a* for any integer *a*. The numerator will always be divisible by 3 so it cannot be a power of 2.

3. (12 points) Simplify the following expressions:

a. $3x^2-5x+7$ when x = -5. $3(-5)^2-5(-5)+7=75+25+7=107$.

b.

$$\left(\frac{4x^4y^3}{12x^{-5}y^6}\right)^{-3}=\left(\frac{4}{12}\right)^{-3}\left(\frac{x^4}{x^{-5}}\right)^{-3}\left(\frac{y^3}{y^6}\right)^{-3}=\left(\frac{1}{3}\right)^{-3}\left(x^9\right)^{-3}\left(\frac{1}{y^3}\right)^{-3}=27x^{-27}y^9 \text{ or } \frac{27y^9}{x^{27}}$$

c. $2(\log_2 5)-(\log_2 200)=\log_2\left(\frac{5^2}{200}\right)=\log_2\left(\frac{25}{200}\right)=\log_2\left(\frac{1}{8}\right)=\log_2\left(2^{-3}\right)=-3$

d.

$\left\lfloor \log_2(100!)-\log_2(99!)\right\rfloor=\left\lfloor \log_2\left(\frac{100!}{99!}\right)\right\rfloor=\left\lfloor \log_2(100)\right\rfloor=6$, as 64 = $2^6<100<2^7=128$.

4. (10 points) Find the roots of each polynomial

a. $3x^2-36x+33=3\left(x^2-12x+11\right)=3(x-1)(x-11)$. The roots are 1 and 11.

b. $x\left(x(x-5)+6\right)=x\left(x^2-5x+6\right)=x(x-2)(x-3)$. The roots 1re 0, 2, and 3.

c. For what values of *t* will $x^2-2x+3t$ have 2 different real roots?

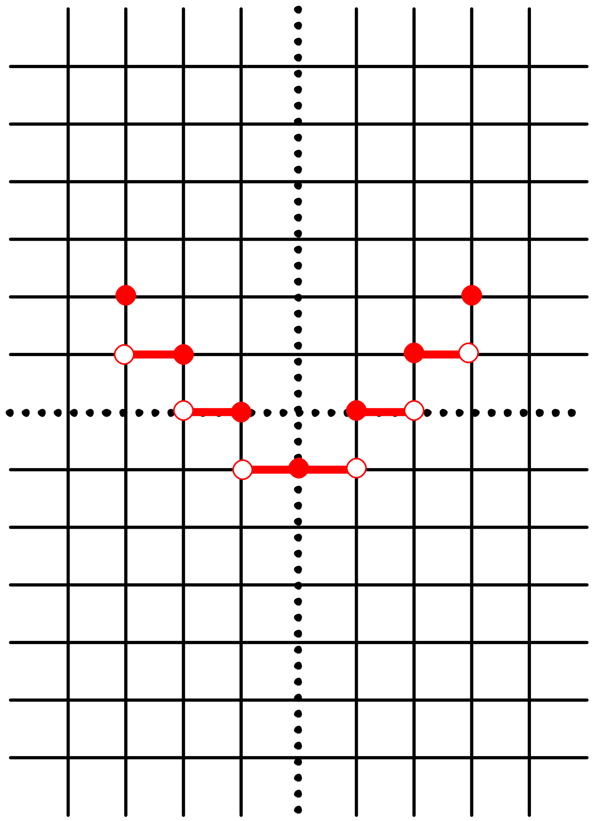
$x^2-2x+3t$ will have two different real roots when the discriminant b^2-4ac is positive. *a* = 1, *b* = -2, and *c* = 3*t*.

$$b^2-4ac=(-2)^2-12t=4-12t. \quad 4-12t>0 \text{ when } \frac{1}{3}>t.$$

5. (10 points) a. Fill in this table:

x	$y=\lfloor abs(x)-1\rfloor$
-3	2
-2	1
-1	0
0	-1
1	0
2	1
3	2

b. Draw a neat sketch of the graph of the function $y = \lfloor abs(x) - 1 \rfloor$ for $-3 \le x \le 3$. Use solid dots and open circles to show what happens at the endpoints.



6. (12 points) Find the prime factorization of each of these integers. Write your answer in exponential form with the prime factors in increasing order, e.g. $12 = 2^2 \cdot 3$.

a. 129 = 3*43

b. $1400 = 14*10*10 = 2*7*2*5*2*5 = 2^3*5^2*7$

c. $11! = 11*10*9*8*7*6*5*4*3*2 = 11*2*5*3^2*2^3*7*2*3*5*2^2*3*2$
$= 2^8*3^4*5^2*7*11$

7. (18 points) Find the gcd or lcm, as indicated.

a. gcd(70, 42) = 14

b. lcm(70, 42) = 210

c. gcd(256, 397) = 1

d. lcm(32, 20) = 160

d. gcd(10!, 5!) = 5! = 120 **f.** lcm(6!, 4!) = 6! = 720

8. (15 points) Evaluate these quantities.

a. 29 **mod** 5 = 4 b. -29 **mod** 5 = 1

c. 5 **mod** 29 = 5 d. 121 **mod** 21 = 16

e. 987654321 **mod** 100000 = 54321

9. (5 points) If a and b are integers with $a \neq 0$, $a \mid b$ is read "*a divides b*" and is true if and only if there is an integer k such that $b = ka$. Use the definition of $a \mid b$ and the field axioms (except for the multiplicative inverse axiom which doesn't hold for $\mathbb{Z}$) to prove that

If a, b and c are integers such that $a \mid b$ then $a \mid bc$.

Proof:
Since $a \mid b$, there is an integer k such that $b = ka$.
Multiplying both sides of this equation by c gives $cb = cka$.
By commutativity, $bc = cka$.
The integers are closed under multiplication so ck is an integer and by the definition of divides, $a \mid bc$.

Q.E.D.