1. (10 points) For each set, give an example of a real number that is in the set and an example of a real number that is not in the set.

<table>
<thead>
<tr>
<th>Set</th>
<th>real number in the set</th>
<th>real number not in the set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}$</td>
<td>3</td>
<td>1/3</td>
</tr>
<tr>
<td>$\mathbb{R}^-$</td>
<td>-3</td>
<td>3</td>
</tr>
<tr>
<td>$\mathbb{Q}$</td>
<td>3</td>
<td>$\sqrt{3}$</td>
</tr>
<tr>
<td>irrationals</td>
<td>$\sqrt{3}$</td>
<td>3</td>
</tr>
<tr>
<td>transcendentals</td>
<td>$\pi$</td>
<td>$\sqrt{3}$</td>
</tr>
</tbody>
</table>

2. (8 points) Which of the following sets of numbers are fields? For those that are not fields, tell all the field axioms that do not hold.

a. $\mathbb{Q}^+$ (the set of all positive numbers in $\mathbb{Q}$)

Not a field. $\mathbb{Q}^+$ does not contain an additive identity (0 is not in $\mathbb{Q}^+$) and does not contain additive inverses (-3 is not in $\mathbb{Q}^+$).

$\mathbb{Q}^+$ is closed under + and $\times$. The commutative, associative, and distributive laws hold as this is just regular real arithmetic.

$\mathbb{Q}^+$ has a multiplicative identity 1 and every element in $\mathbb{Q}^+$ has a multiplicative inverse in $\mathbb{Q}^+$.

b. The set of all rationals $\frac{a}{b}$ where $a$ and $b$ are integers and $b$ is a power of 2.

This set is closed under addition and multiplication,

$$\frac{a}{2^m} + \frac{b}{2^n} = \frac{a2^n + b2^m}{2^{m+n}} \quad \text{and} \quad \frac{a}{2^m} \cdot \frac{b}{2^n} = \frac{ab}{2^{m+n}}$$

The commutative, associative, and distributive laws hold as this is just regular real arithmetic.

$0 = \frac{0}{2^0}$ is the additive identity. $1 = \frac{1}{2^0}$ is the multiplicative identity.

Every element $\frac{a}{2^i}$ in this set has an additive inverse in the set $\frac{-a}{2^i}$.

But, not every element in this set has a multiplicative inverse in this set. The multiplicative inverse of 3/2 is 2/3 or $2a / 3a$ for any non-zero integer $a$. The denominator will never be a power of 2.
3. (12 points) Simplify the following expressions:

a. \(3x^2 - 5x + 7\) when \(x = -4\). \(3(-4)^2 - 5(-4) + 7 = 48 + 20 + 7 = 75\)

b. \(\left(\frac{3x^4 - 3}{12x^5 y^6}\right)^{-3} = \left(\frac{3}{12}\right)^{-3} \left(\frac{x^4}{x^5}\right)^{-3} \left(\frac{y^{-3}}{y^6}\right)^{-3} = \left(\frac{1}{4}\right)^{-3} \left(\frac{1}{x}\right)^{-3} \left(\frac{1}{y^3}\right)^{-3} = 4^3 x^3 y^{27}\)

c. \(2(\log_2 3) - (\log_2 36) = \log_2 \left(\frac{9}{36}\right) = \log_2 \left(\frac{1}{4}\right) = \log_2 (2^{-2}) = -2\)

d. \([ \log_2 (1000!) - \log_2 (999!) ] = [ \log_2 \left(\frac{1000!}{999!}\right) ] = [ \log_2 (1000) ] = 9\) since \(2^9 = 512 < 1000 < 1024 = 2^{10}\).

4. (10 points) Find the roots of each polynomial

a. \(2x^2 - 18x + 28 = 2(x^2 - 9x + 14) = 2(x - 2)(x - 7)\)

The roots are 2 and 7.

b. \(x(x(x - 3) + 2) = x(x^2 - 3x + 2) = x(x - 1)(x - 2)\)

The roots are 0, 1, and 2.

c. For what values of \(t\) will \(x^2 - 2tx + 3\) have 2 different real roots?

\(x^2 - 2tx + 3\) will have 2 different real roots when the discriminant \(b^2 - 4ac\) is positive. \(a = 1, b = -2t, c = 3\).

\(b^2 - 4ac = 4t^2 - 12\) which is positive when \(4t^2 > 12\) or \(t^2 > 3\).

\(t^2 > 3\) is true when \(t > \sqrt{3}\) or when \(t < -\sqrt{3}\).

5. (10 points) a. Fill in this table:

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y = \text{abs} \left(\lceil x \rceil \right) + \lceil x \rceil)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>
b. Draw a neat sketch of the graph of the function \( y = \text{abs} \left( \lfloor x \rfloor \right) + \left\lfloor x \right\rfloor \) for \(-3 \leq x \leq 3\). Use solid dots and open circles to show what happens at the endpoints.

6. (12 points) Find the prime factorization of each of these integers. Write your answer in exponential form with the prime factors in increasing order, e.g. 12 = \( 2^2 \cdot 3 \).

   a. \( 102 = 2 \cdot 3 \cdot 17 \)
   
   b. \( 18000 = 2^4 \cdot 3^2 \cdot 5^3 \)
   
   c. \( 12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \)

7. (18 points) Find the gcd or lcm, as indicated.

   a. \( \text{gcd}(28, 42) = 14 \)
   
   b. \( \text{lcm}(28, 42) = 84 \)
   
   c. \( \text{gcd}(256, 397) = 1 \)
   
   d. \( \text{lcm}(16, 20) = 80 \)
   
   d. \( \text{gcd}(12!, 6!) = 6! = 720 \)
   
   f. \( \text{lcm}(5!, 3!) = 3! = 6 \)
8. (15 points) Evaluate these quantities.

a. $32 \mod 5 = 2$

b. $-32 \mod 5 = 3$

c. $5 \mod 27 = 5$

d. $111 \mod 21 = 6$

e. $987654321 \mod 10000 = 4321$

9. (5 points) If $a$ and $b$ are integers with $a \neq 0$, $a \mid b$ is read "$a$ divides $b$" and is true if and only if there is an integer $k$ such that $b = ka$. Use the definition of $a \mid b$ and the field axioms (except for the multiplicative inverse axiom which doesn't hold for $\mathbb{Z}$) to prove that

If $a$, $b$, and $c$ are integers such that $a \mid b$ and $b \mid c$ then $a \mid c$.

Proof:
Since $a \mid b$ there is an integer $m$ such that $b = ma$.

Since $b \mid c$ there is an integer $n$ such that $c = nb$.

Substituting equals for equals $c = n ma$.

$nm$ is an integer since the integers are closed under multiplication.

Therefore, by the definition of "divides," $a \mid c$.

Q.E.D.