**SOLUTION**

**College of Computer and Information Science** **CS U200 Discrete Structures**
**Professor Harriet Fell** **First Hour Exam – Section 1** **September 29, 2004**

**1. (10 points)** For each set, give an example of a real number that is in the set and an example of a real number that is not in the set.

| Set | real number in the set | real number not in the set |
|---|---|---|
| $\mathbb{Z}$ | 3 | 1/3 |
| $\mathbb{R}^-$ | -3 | 3 |
| $\mathbb{Q}$ | 3 | $\sqrt{3}$ |
| **irrationals** | $\sqrt{3}$ | 3 |
| **transcendentals** | $\pi$ | $\sqrt{3}$ |

**2. (8 points)** Which of the following sets of numbers are fields? For those that are not fields, tell all the field axioms that do not hold.

**a.** $\mathbb{Q}^+$ (the set of all positive numbers in $\mathbb{Q}$)

Not a field. $\mathbb{Q}^+$ does not contain an additive identity (0 is not in $\mathbb{Q}^+$**)** and does not contain additive inverses (-3 is not in $\mathbb{Q}^+$**)**.

$\mathbb{Q}^+$ is closed under + and $\times$. The commutative, associative, and distributive laws hold as this is just regular real arithmetic.

$\mathbb{Q}^+$ has a multiplicative identity 1 and every element in $\mathbb{Q}^+$ has a multiplicative inverse in $\mathbb{Q}^+$.

**b.** The set of all rationals $\frac{a}{b}$ where $a$ and $b$ are integers and $b$ is a power of 2.

This set is closed under addition and multiplication,

$$\frac{a}{2^m}+\frac{b}{2^n}=\frac{a2^n+b2^m}{2^{m+n}} \qquad \frac{a}{2^m}\bullet\frac{b}{2^n}=\frac{ab}{2^{m+n}}$$

The commutative, associative, and distributive laws hold as this is just regular real arithmetic.

$0=\frac{0}{2^0}$ is the additive identity. $1=\frac{1}{2^0}$ is the multiplicative identity.

Every element $\frac{a}{2^k}$ in this set has an additive inverse in the set, $\frac{-a}{2^k}$.

But, not every element in this set has a multiplicative inverse in this set. The multiplicative inverse of 3/2 is 2/3 or $2a/3a$ for any non-zero integer $a$. The denominator will never be a power of 2.

**3. (12 points)** Simplify the following expressions:

**a.** $3x^2 - 5x + 7$ when $x$ = *-4*. $3(-4)^2 - 5(-4) + 7 = 48 + 20 + 7 = 75$

**b.**

$$\left(\frac{3x^4y^{-3}}{12x^5y^6}\right)^{-3} = \left(\frac{3}{12}\right)^{-3}\left(\frac{x^4}{x^5}\right)^{-3}\left(\frac{y^{-3}}{y^6}\right)^{-3} = \left(\frac{1}{4}\right)^{-3}\left(\frac{1}{x}\right)^{-3}\left(y^{-9}\right)^{-3} = 4^3x^3y^{27}$$

**c.** $2(\log_2 3) - (\log_2 36) = \log_2\left(\frac{3^2}{36}\right) = \log_2\left(\frac{9}{36}\right) = \log_2\left(\frac{1}{4}\right) = \log_2\left(2^{-2}\right) = -2$

**d.** $\lfloor \log_2(1000!) - \log_2(999!) \rfloor = \left\lfloor \log_2\left(\frac{1000!}{999!}\right)\right\rfloor = \lfloor \log_2(1000) \rfloor = 9$

since $2^9 = 512 < 1000 < 1024 = 2^{10}$.

**4. (10 points)** Find the roots of each polynomial

**a.** $2x^2 - 18x + 28 = 2\left(x^2 - 9x + 14\right) = 2(x-2)(x-7)$

The roots are 2 and 7.

**b.** $x\left(x(x-3)+2\right) = x\left(x^2 - 3x + 2\right) = x(x-1)(x-2)$

The roots are 0, 1, and 2.

**c.** For what values of $t$ will $x^2 - 2tx + 3$ have 2 different real roots?

$x^2 - 2tx + 3$ will have 2 different real roots when the discriminant $b^2 - 4ac$ is positive.
$a$ = 1, $b$=-2t, and $c$ = 3.

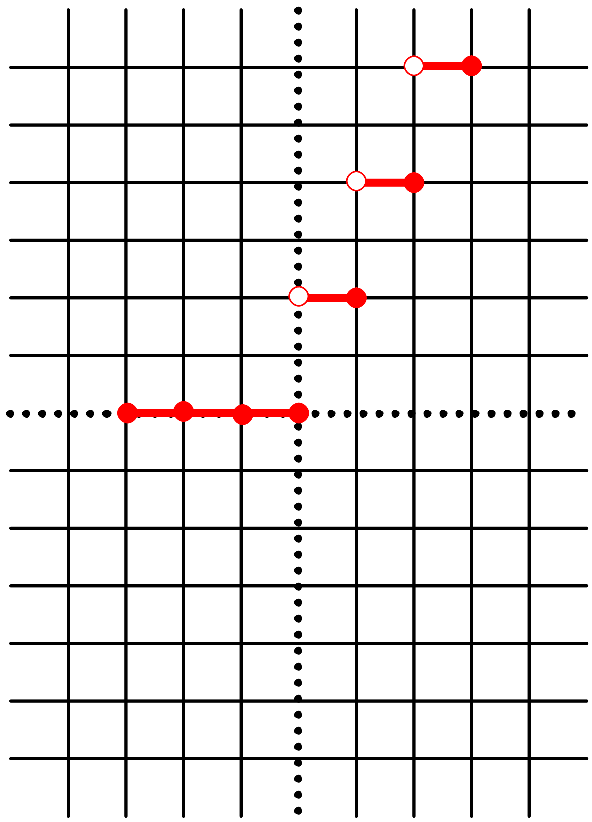
$b^2 - 4ac = 4t^2 - 12$ which is positive when $4t^2 > 12$ or $t^2 > 3$.

$t^2 > 3$ is true when t >$\sqrt{3}$ or when t <-$\sqrt{3}$.

**5. (10 points) a.** Fill in this table:

| x | $y = abs(\lceil x \rceil) + \lceil x \rceil$ |
|---|---|
| -3 | 0 |
| -2 | 0 |
| -1 | 0 |
| 0 | 0 |
| 1 | 2 |
| 2 | 4 |
| 3 | 6 |

**b.** Draw a neat sketch of the graph of the function $y = abs(\lceil x \rceil) + \lceil x \rceil$ for $-3 \le x \le 3$. Use solid dots and open circles to show what happens at the endpoints.



**6. (12 points)** Find the prime factorization of each of these integers. Write your answer in exponential form with the prime factors in increasing order, e.g. $12 = 2^2 \cdot 3$.

**a.** 102 = 2*3*17

**b.** $18000 = 18*10*10*10 = 2*3^2*2*5*2*5*2*5 = 2^4*3^2*5^3$

**c.** $12! = 12*11*10*9*8*7*6*5*4*3*2 =$
$2^2*3*11*2*5*3^2*2^3*7*2*3*5*2^2*2 = 2^{10}*3^5*5^2*7*11$

**7. (18 points)** Find the gcd or lcm, as indicated.

**a.** gcd(28, 42) = 14

**b.** lcm(28, 42) = 84

**c.** gcd(256, 397) = 1

**d.** lcm(16, 20) = 80

**d.** gcd(12!, 6!) = 6! = 720

**f.** lcm(5!, 3!) = 3! = 6

**8. (15 points)** Evaluate these quantities.

a. 32 **mod** 5 = 2

b. -32 **mod** 5 = 3

c. 5 **mod** 27 = 5

d. 111 **mod** 21 = 6

e. 987654321 **mod** 10000 = 4321

**9. (5 points)** If $a$ and $b$ are integers with $a \neq 0$, $a|b$ is read "*a divides b*" and is true if and only if there is an integer $k$ such that $b = ka$. Use the definition of $a|b$ and the field axioms (except for the multiplicative inverse axiom which doesn't hold for $\mathbb{Z}$) to prove that

If $a, b$ and $c$ are integers such that $a|b$ and $b|c$ then $a|c$.

Proof:
Since $a|b$ there is an integer $m$ such that $b = ma$.
Since $b|c$ there is an integer $n$ such that $c = nb$.
Substituting equals for equals $c = n\,ma$.
$nm$ is an integer since the integers are closed under multiplication.
Therefore, by the definition of "divides," $a|c$.

Q.E.D.