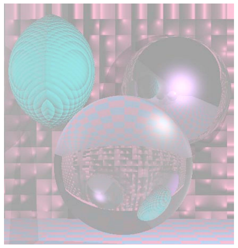


# CS 4300
# Computer Graphics

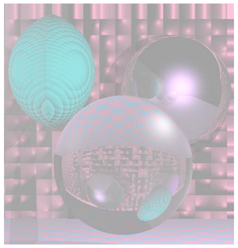
Prof. Harriet Fell

Fall 2012

Lecture 11 – September 27, 2012

# Today's Topics

- Linear Algebra Review
  - Matrices
  - Transformations
- New Linear Algebra
  - Homogeneous Coordinates



# Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \qquad C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

- We use 2x2, 3x3, and 4x4 matrices in computer graphics.
- We'll start with a review of 2D matrices and transformations.

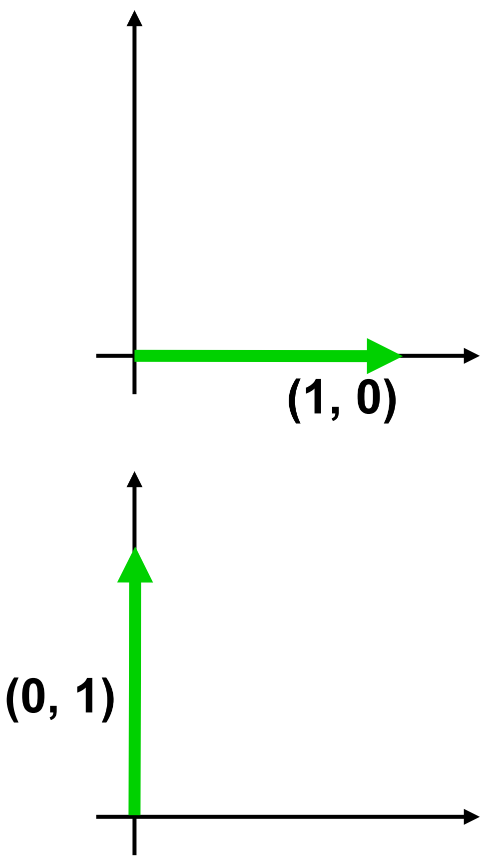
# Basic 2D Linear Transforms

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

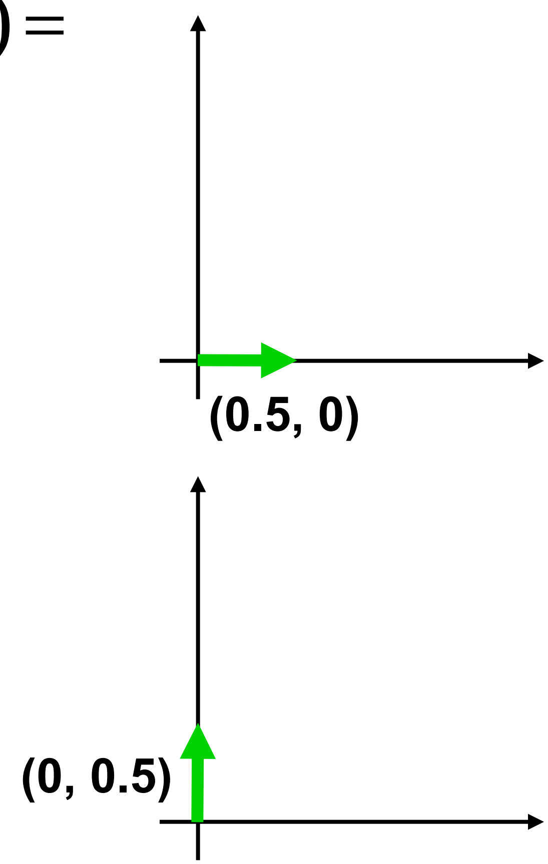
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

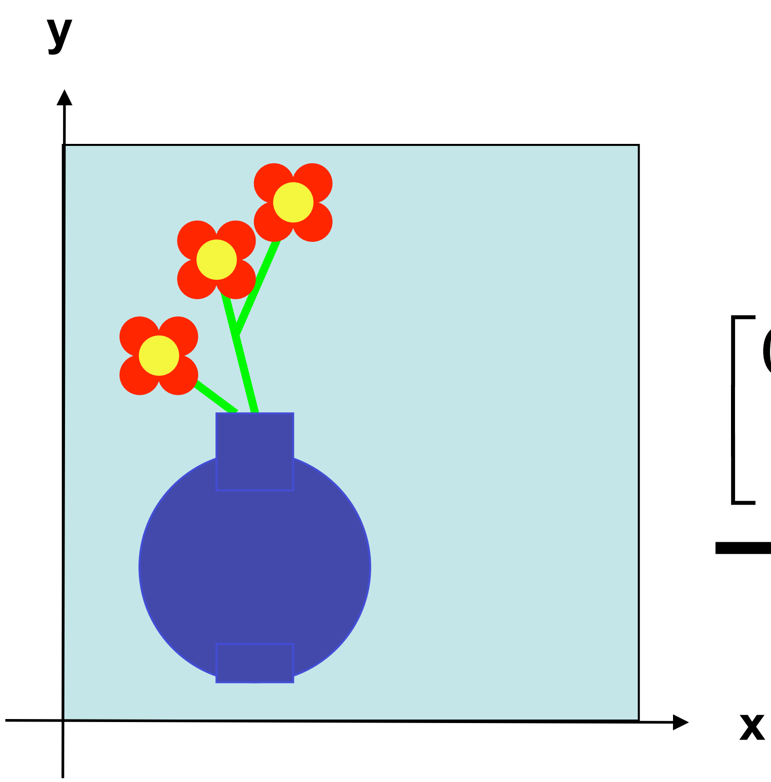
# Scale by .5



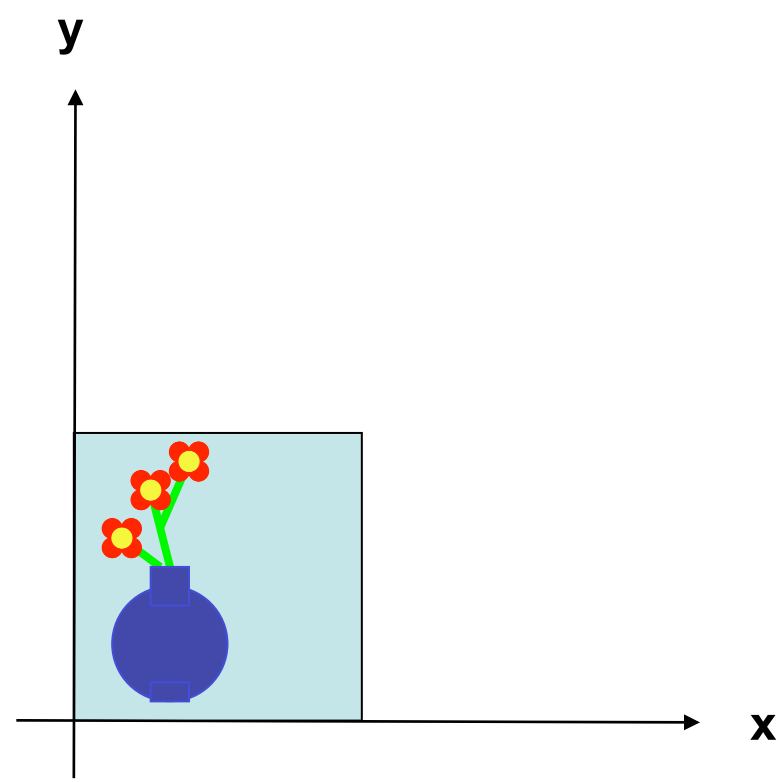

$$\text{scale}(.5, .5) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$



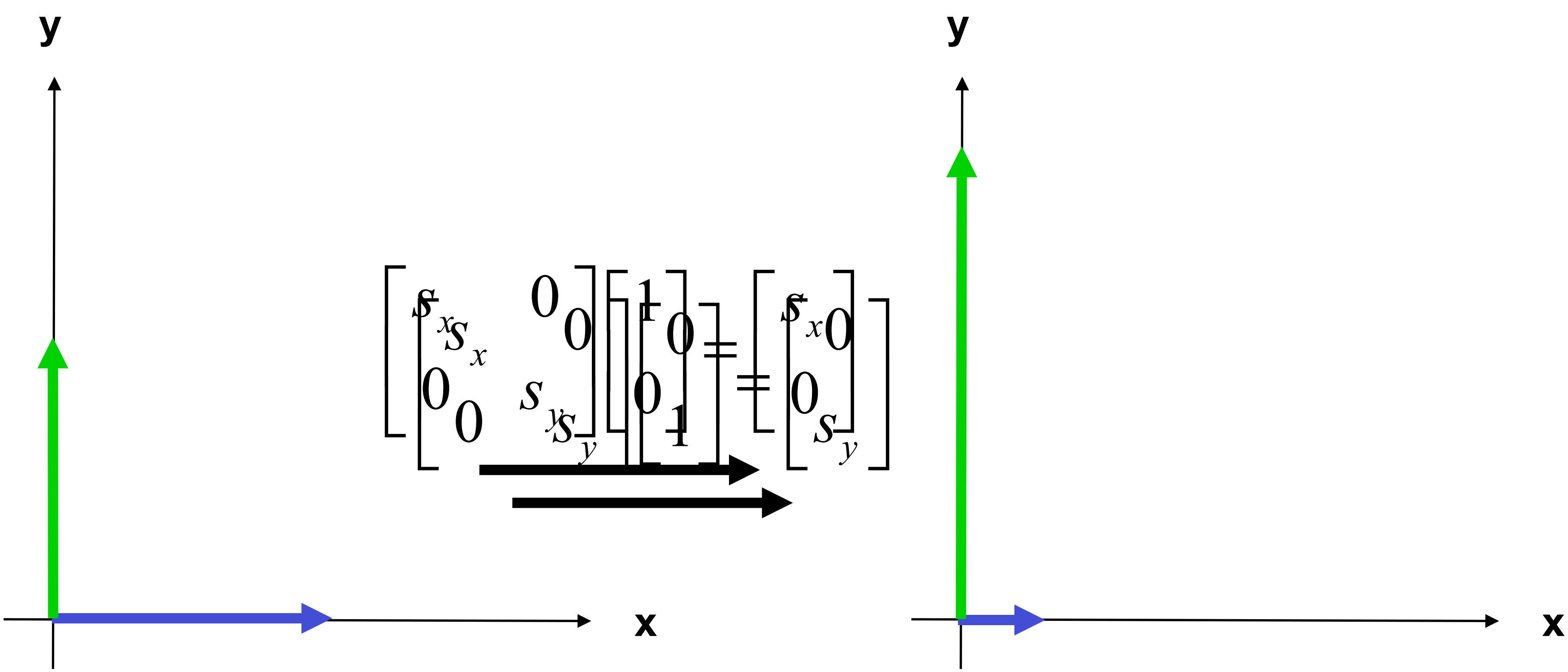
# Scaling by .5



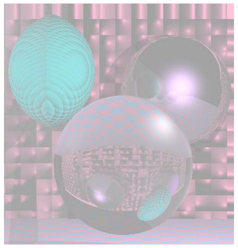

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$



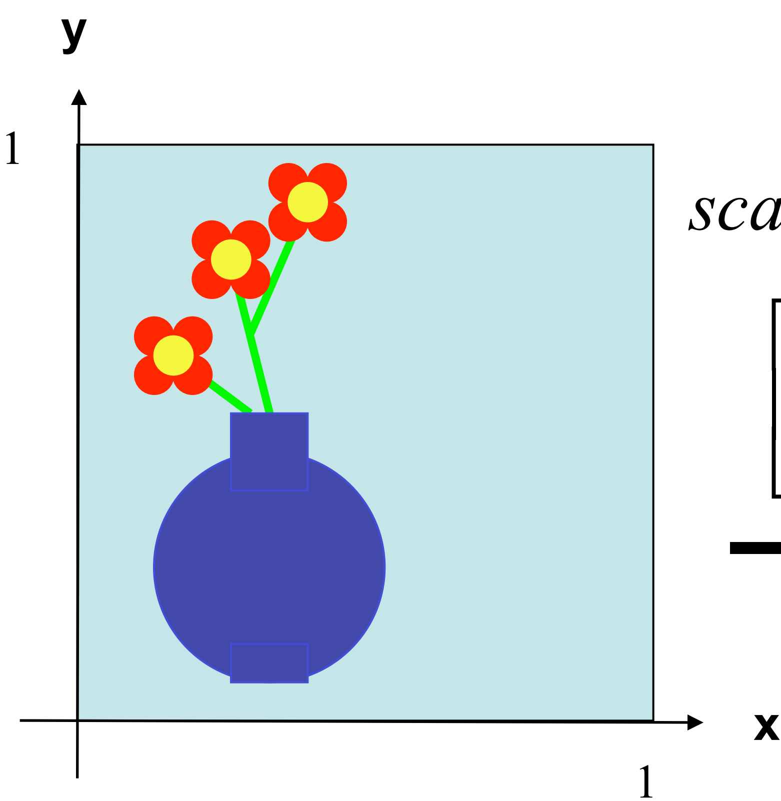
# General Scaling



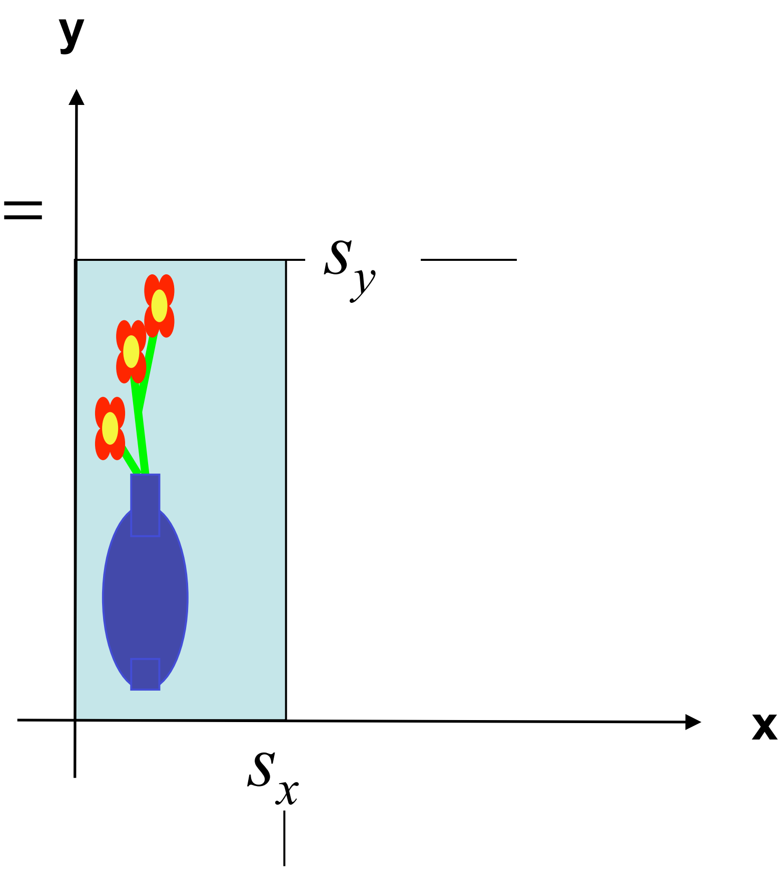

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$



# General Scaling

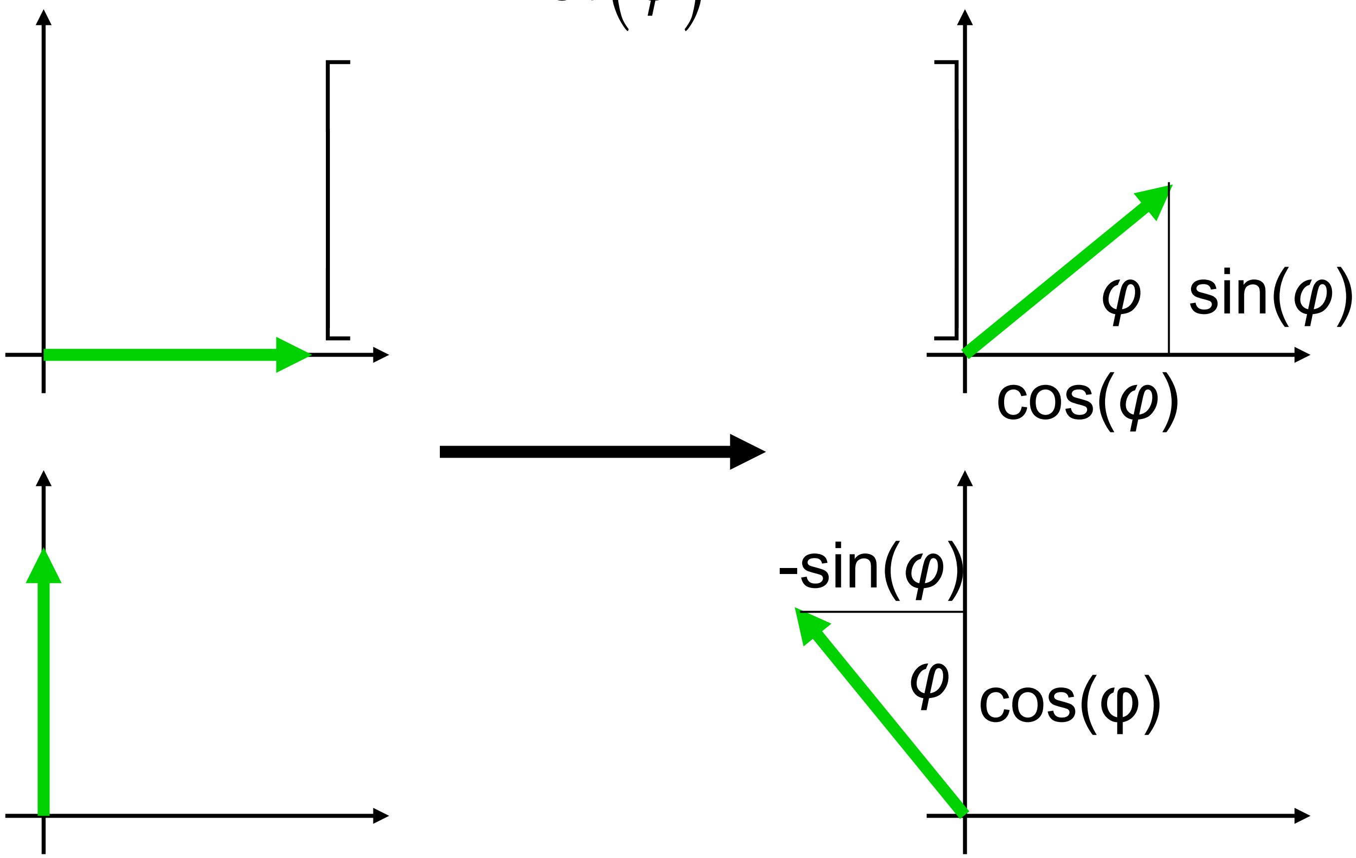



$$scale(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$



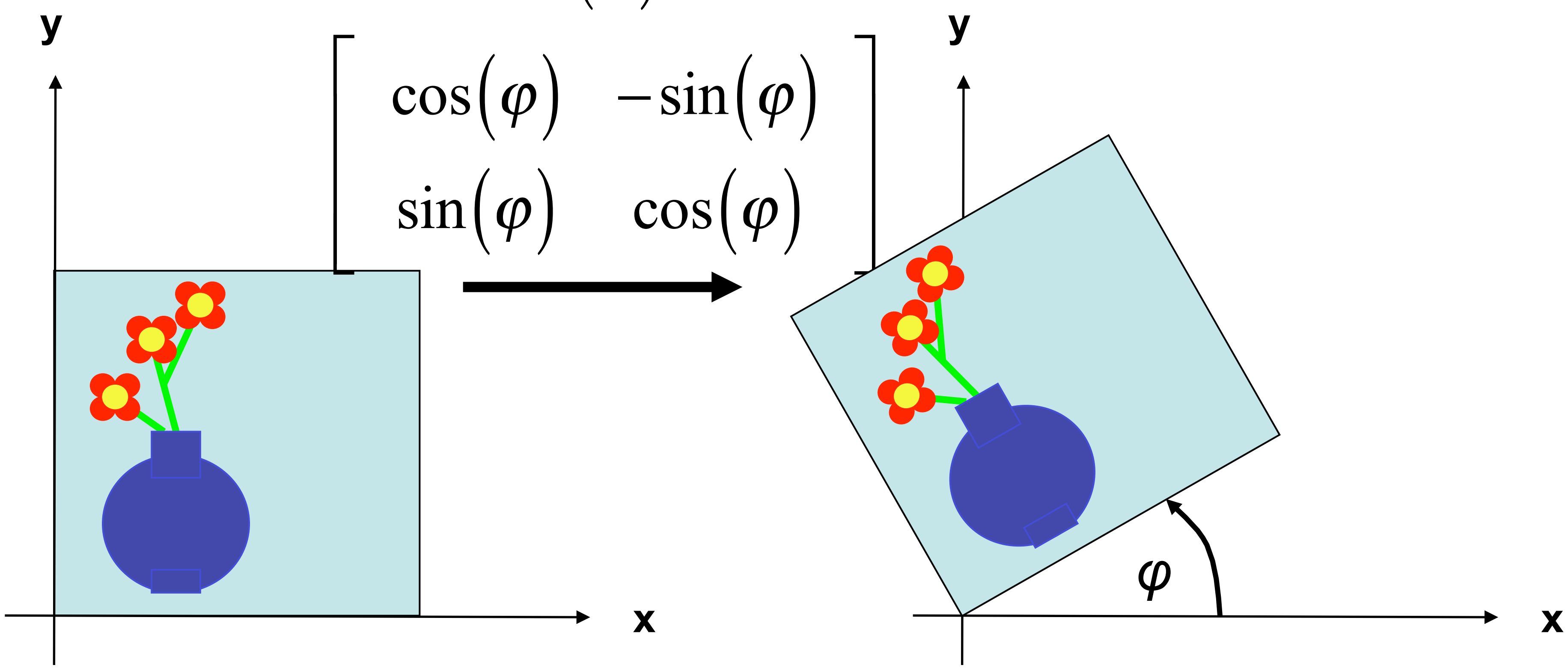
# Rotation

$$\mathrm{rot}(\varphi) =$$

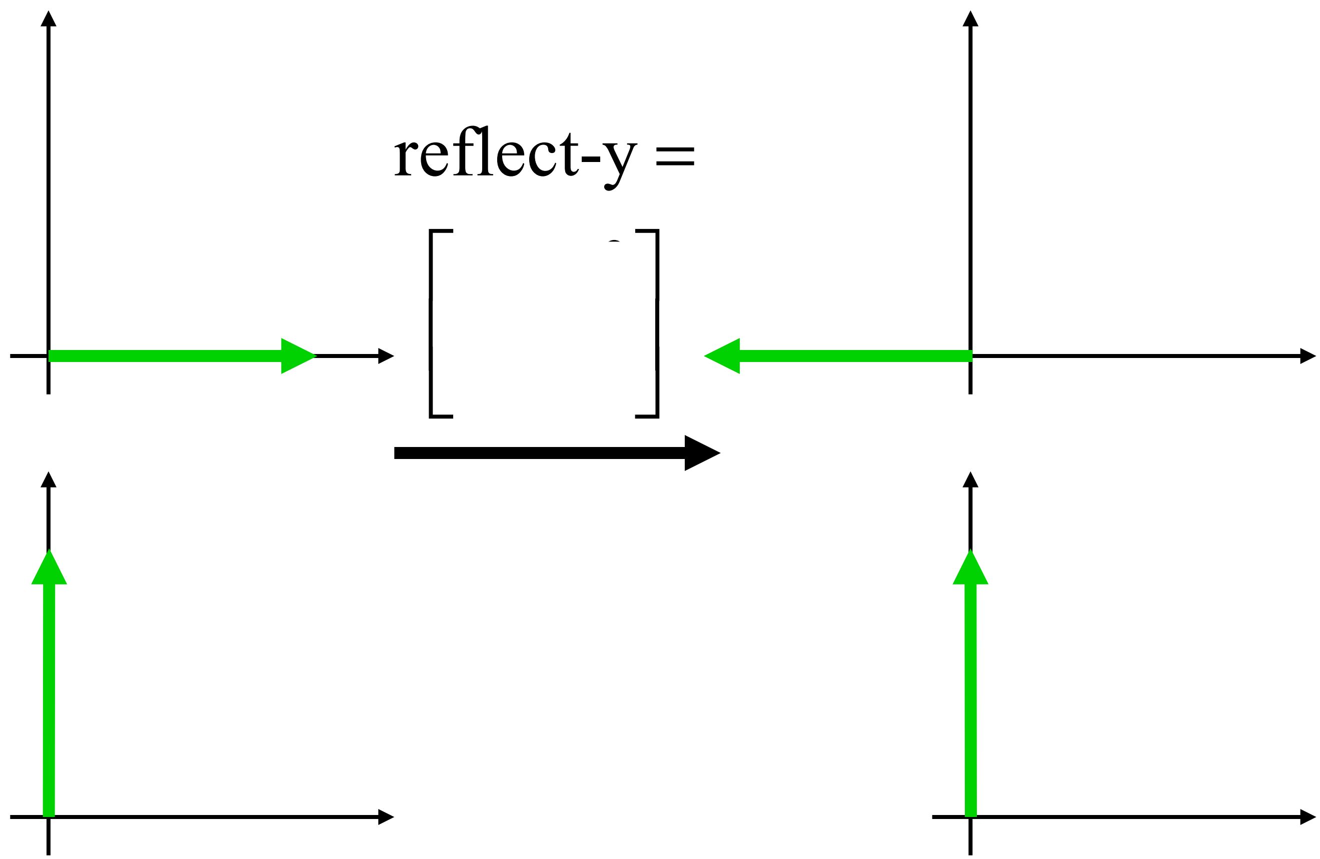


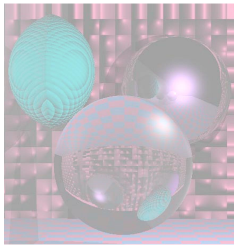
# Rotation

$$\mathrm{rot}(\varphi) = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

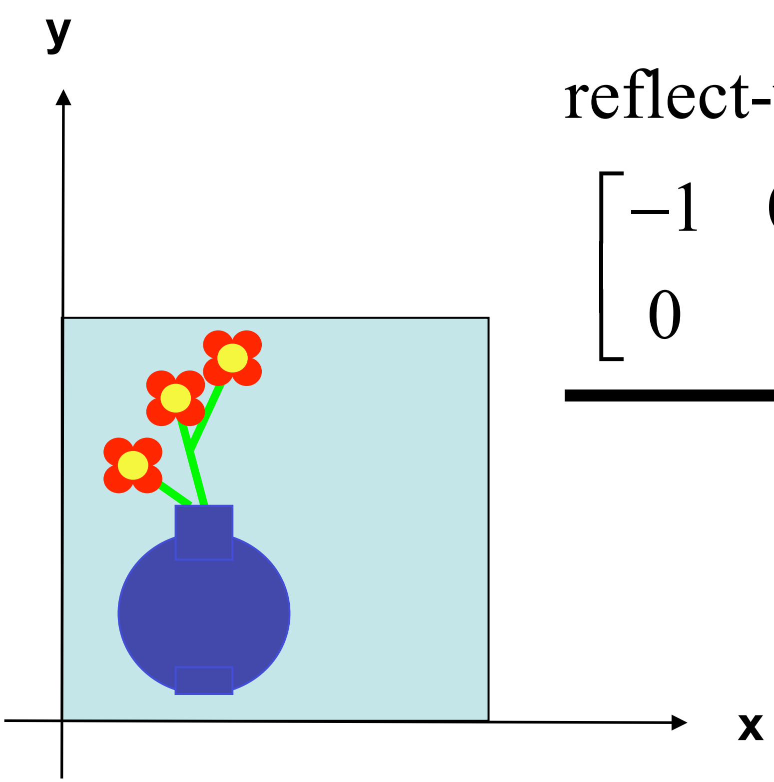


# Reflection in y-axis



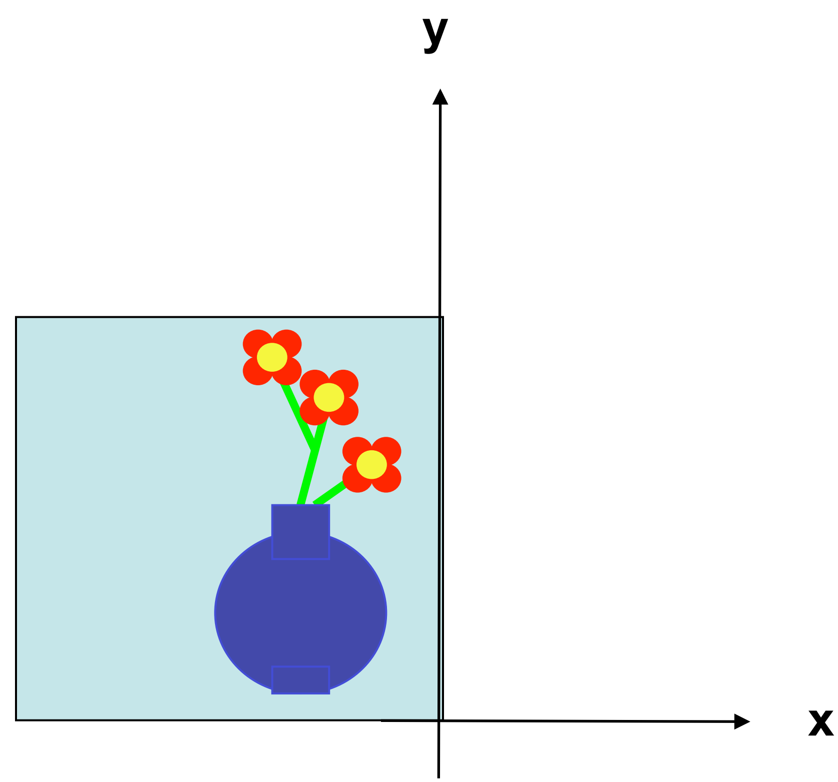


# Reflection in y-axis

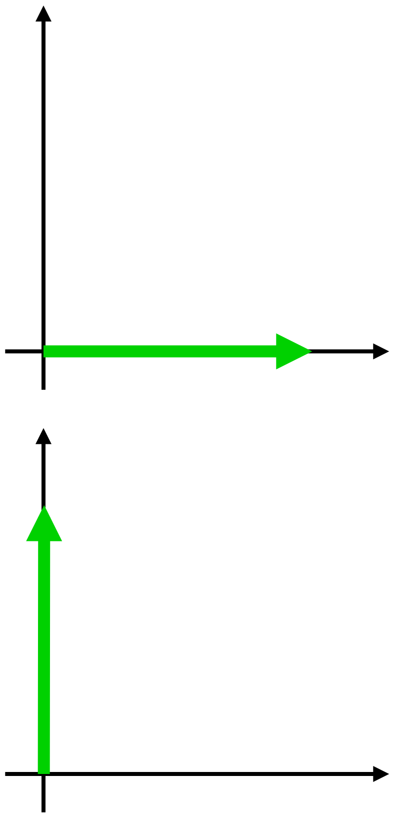



reflect-y =

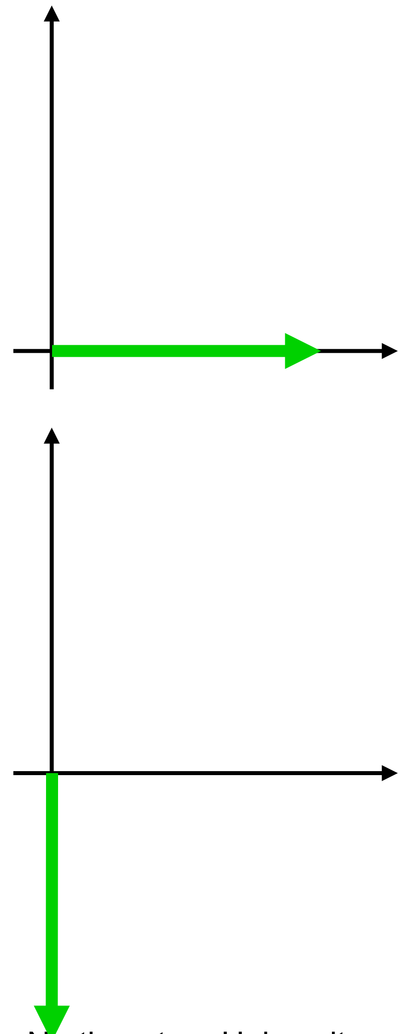
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



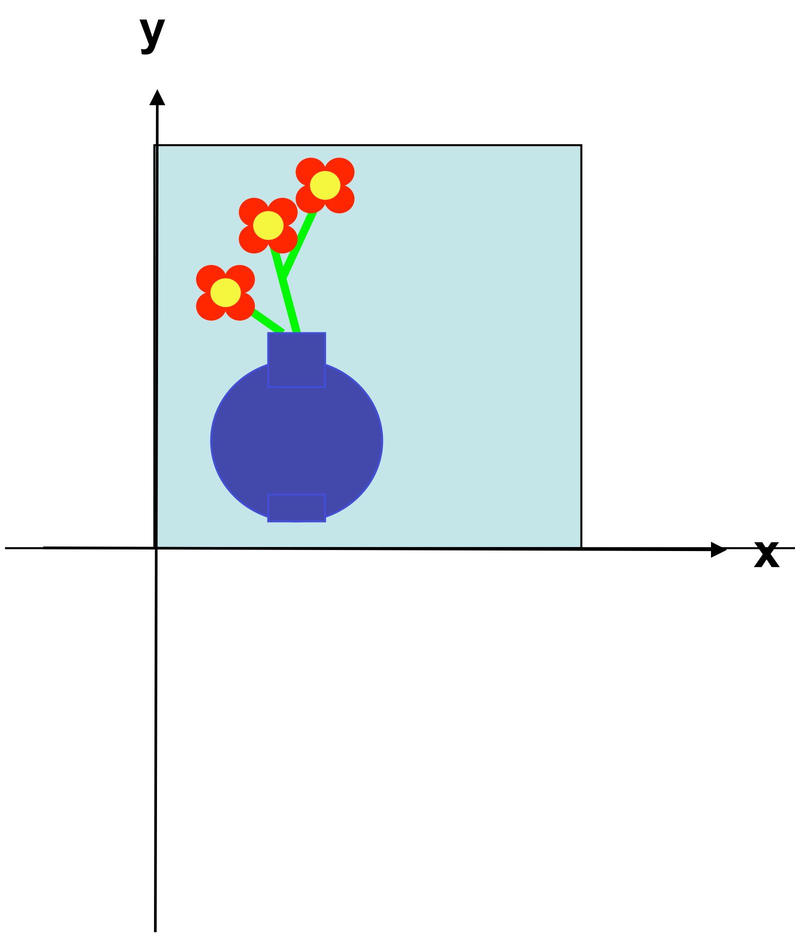
# Reflection in x-axis



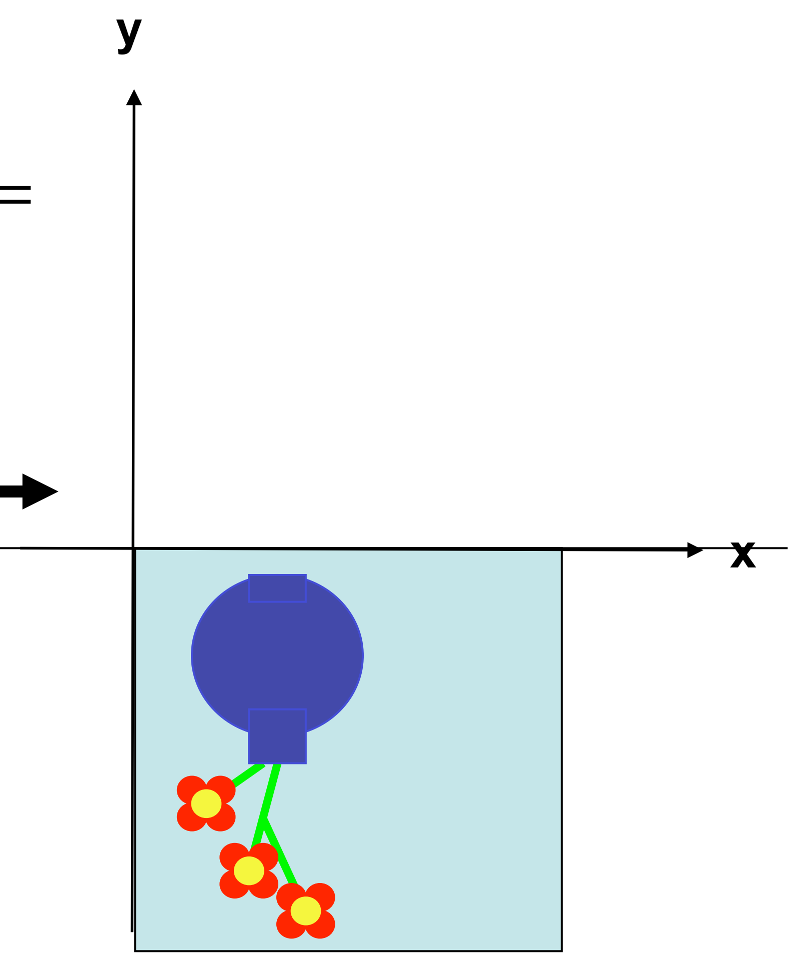
$$\text{reflect-x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



# Reflection in x-axis

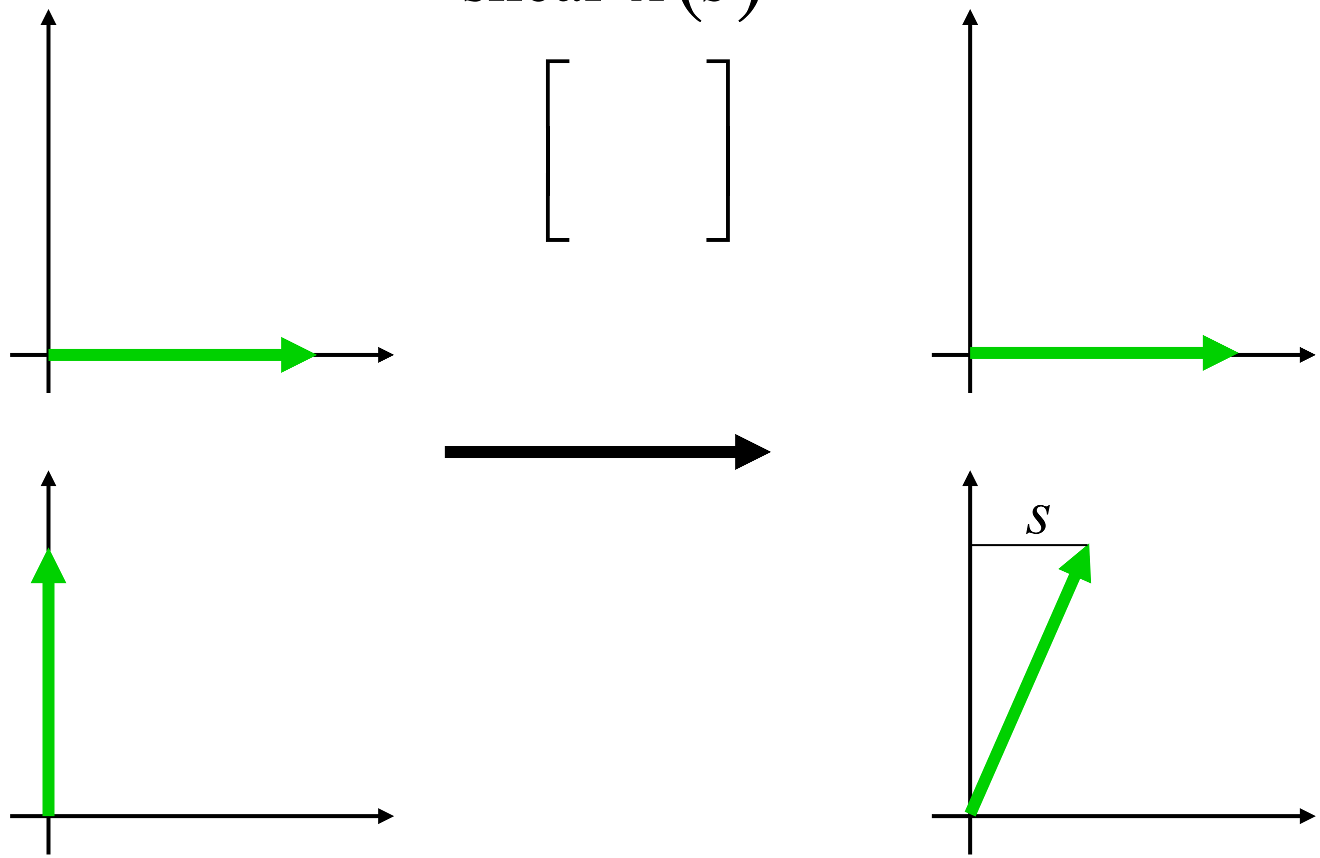



$$\text{reflect-x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

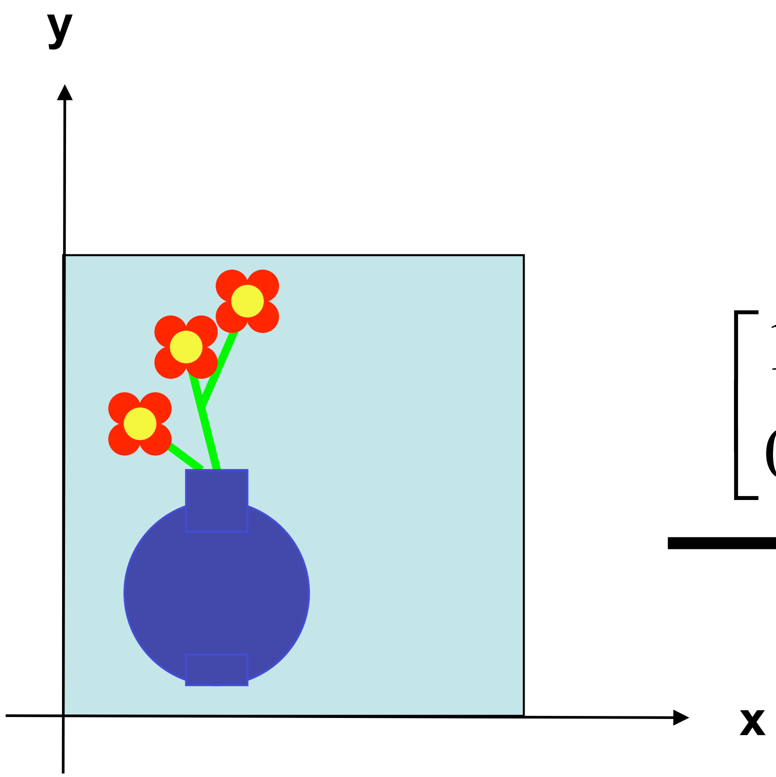


# Shear-x

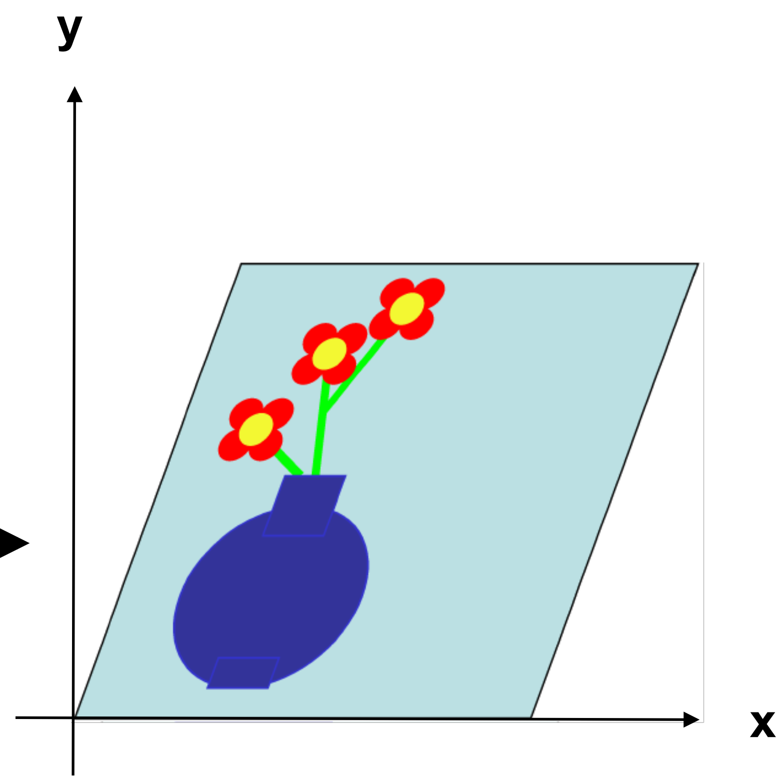
$$\text{shear-x}(s) = $$



# Shear x

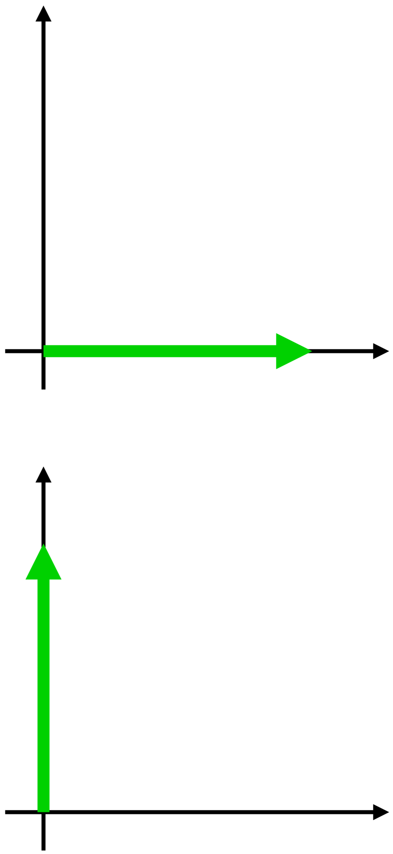


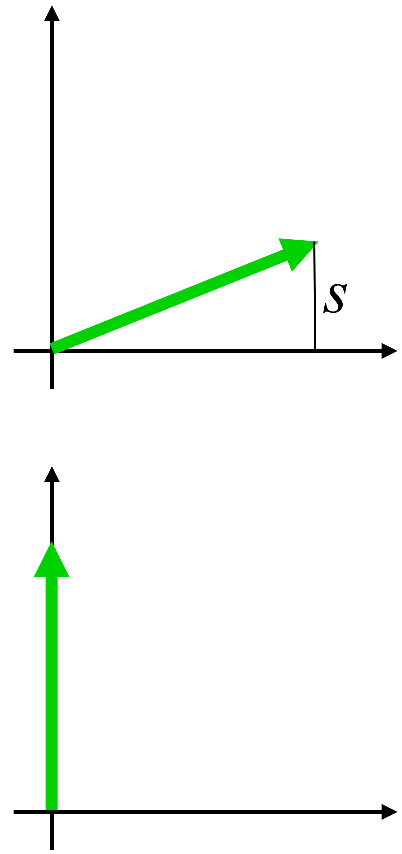

$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$



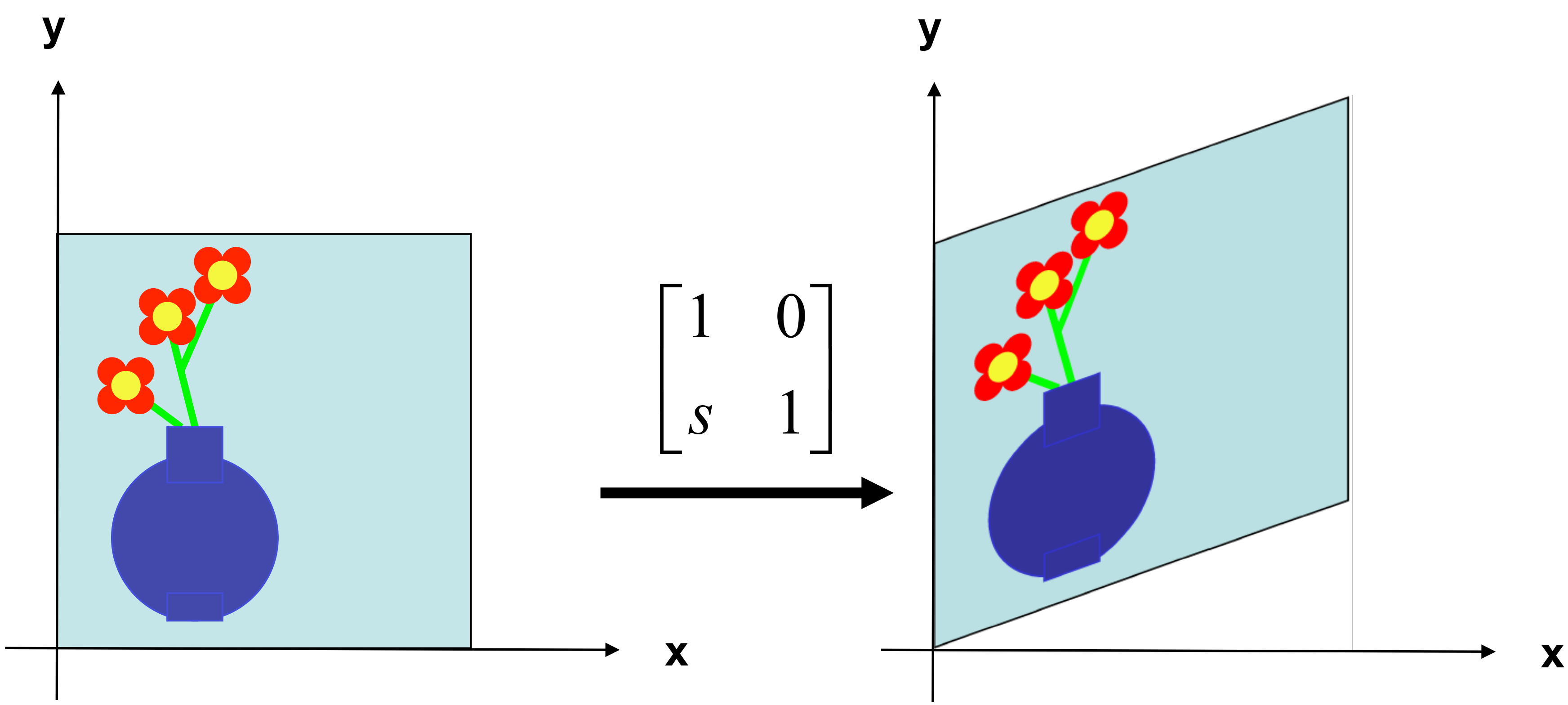
# Shear-y

$$\text{shear-y}(s) = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$



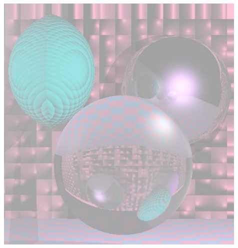


# Shear y




$$\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$



# Linear Transformations

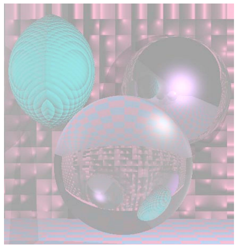
- Scale, Reflection, Rotation, and Shear are all linear transformations
- They satisfy: T(*a***u** + *b***v**) = *a*T(**u**) + *b*T(**v**)
  - **u** and **v** are vectors
  - *a* and *b* are scalars
- If T is a linear transformation
  - T((0, 0)) = (0, 0)

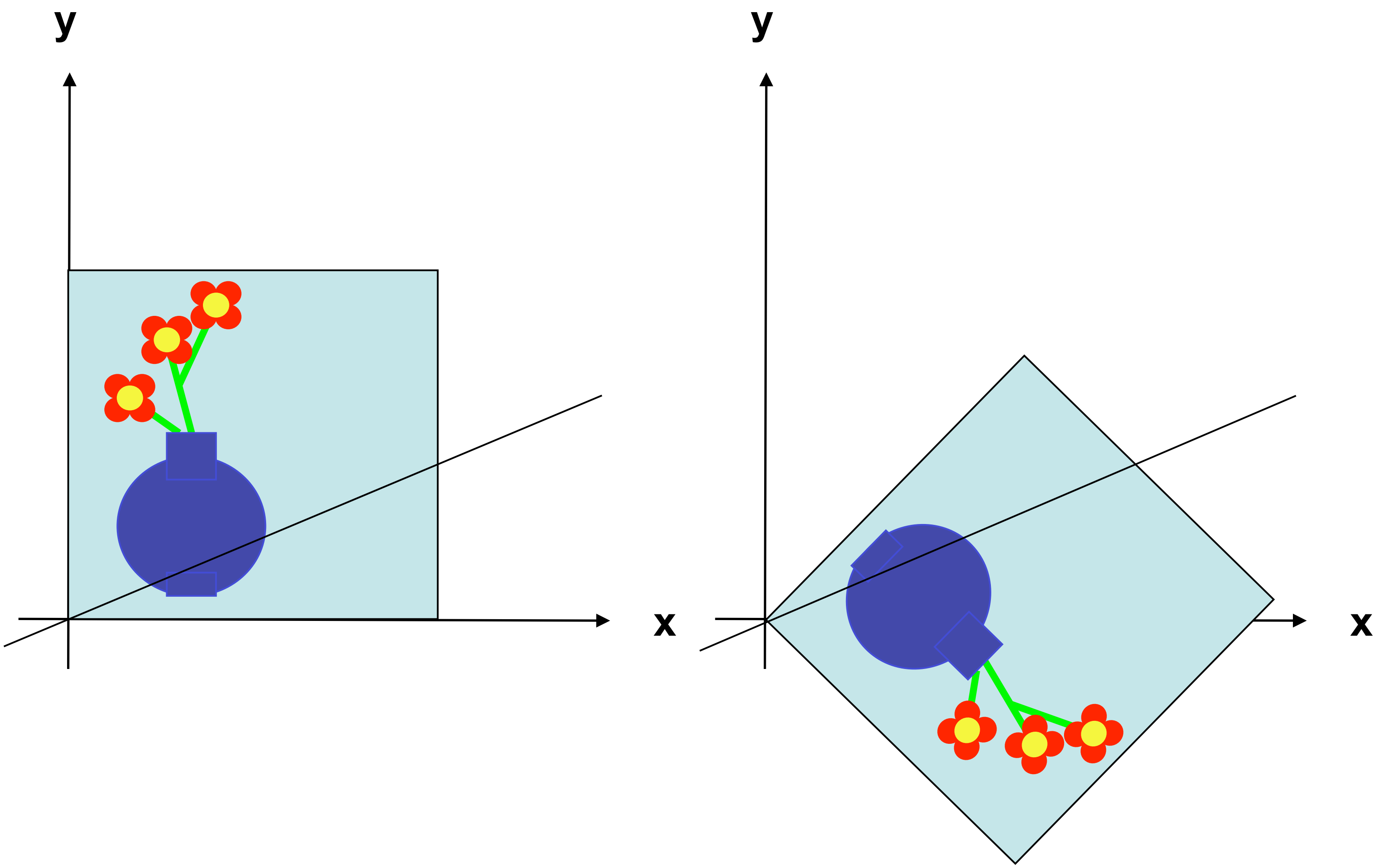
# Composing Linear Transformations

- If $T_1$ and $T_2$ are transformations
  - $T_2 T_1(\mathbf{v}) =_{def} T_2( T_1(\mathbf{v}))$
- If $T_1$ and $T_2$ are linear and are represented by matrices $M_1$ and $M_2$
  - $T_2 T_1$ is represented by $M_2 M_1$
  - $T_2 T_1(\mathbf{v}) = T_2( T_1(\mathbf{v})) = (M_2 M_1)(\mathbf{v})$

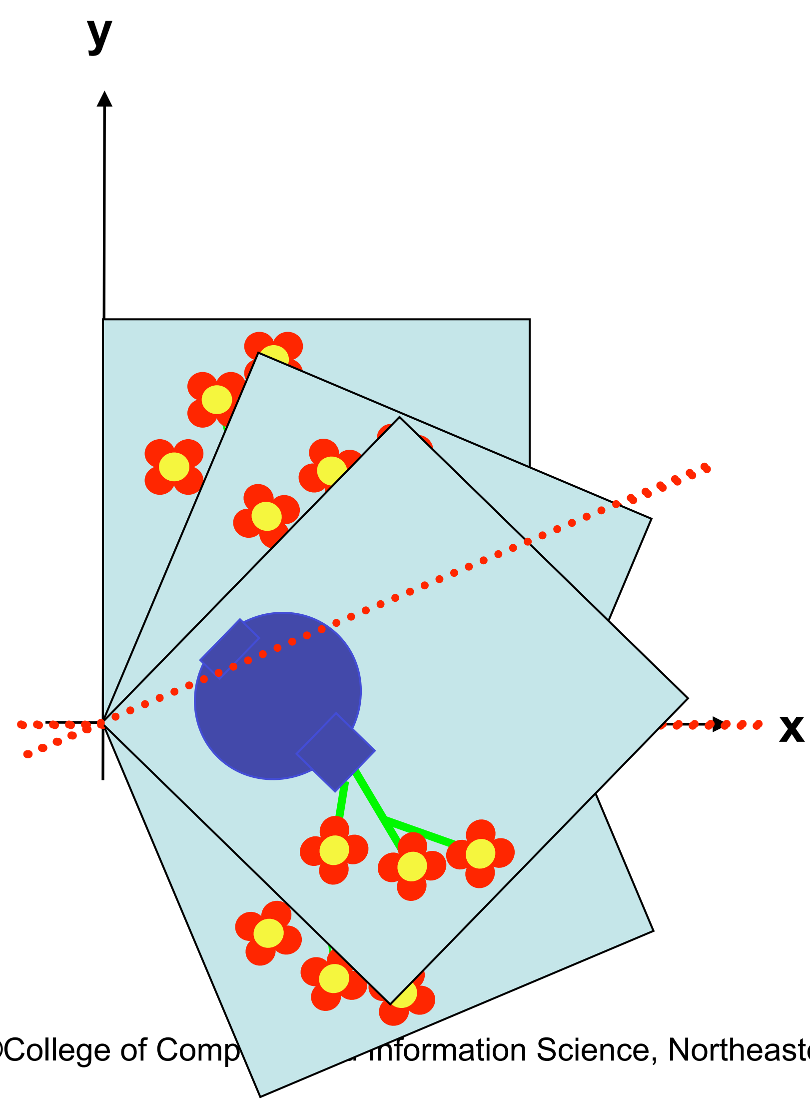
  



# Reflection About an Arbitrary Line (through the origin)



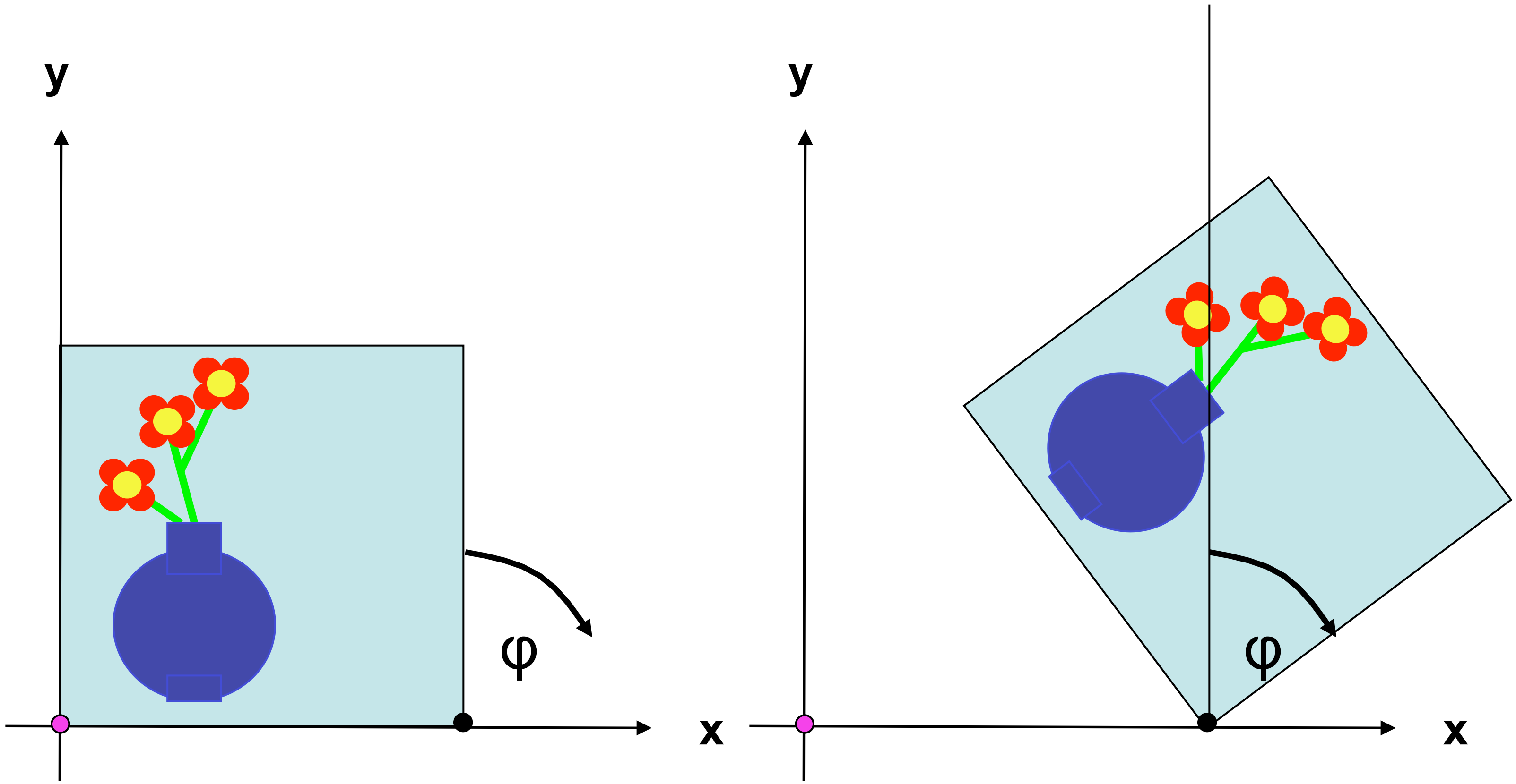
# Reflection as a Composition



# Decomposing Linear Transformations

- Any 2D Linear Transformation can be decomposed into the product of a rotation, a scale, and a rotation if the scale can have negative numbers.
- $M = R_1 S R_2$

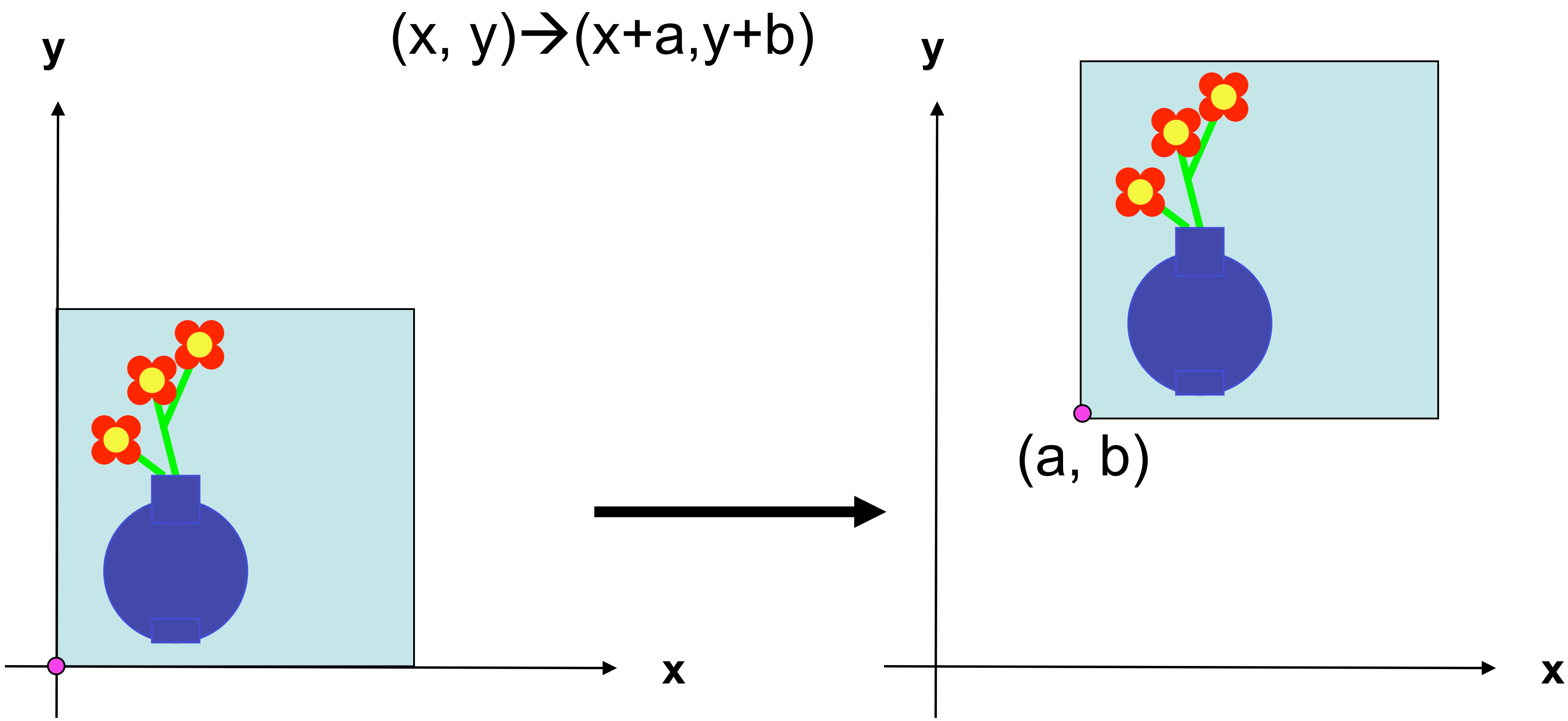
# Rotation about an Arbitrary Point




This is not a linear transformation. The origin moves.

# Translation

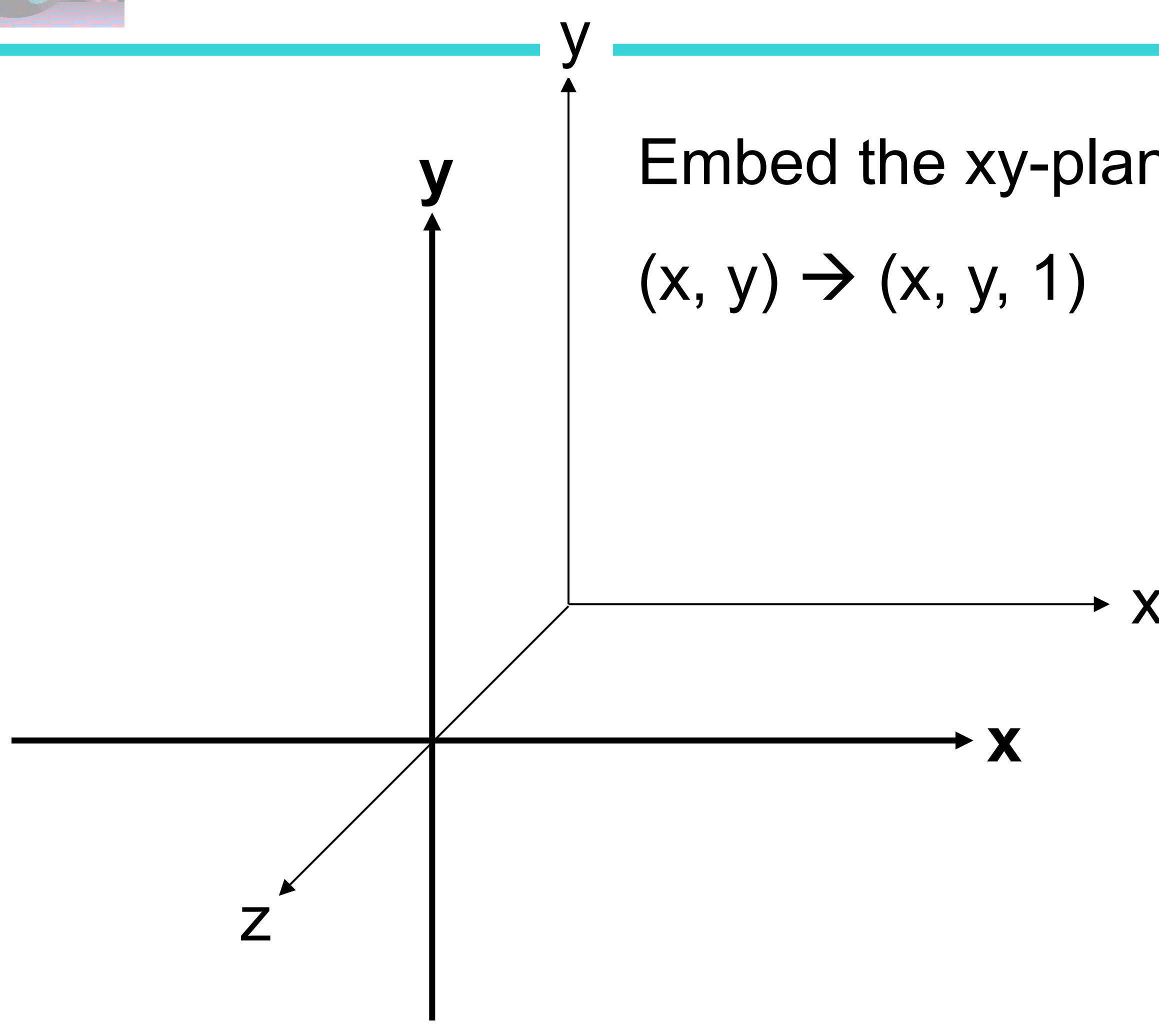



This is not a linear transformation. The origin moves.

# Homogeneous Coordinates

Embed the xy-plane in $R^3$ at $z = 1$.

$(x, y) \rightarrow (x, y, 1)$



# 2D Linear Transformations as 3D Matrices

Any 2D linear transformation can be represented by a 2x2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

or a 3x3 matrix

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \\ 1 \end{bmatrix}$$

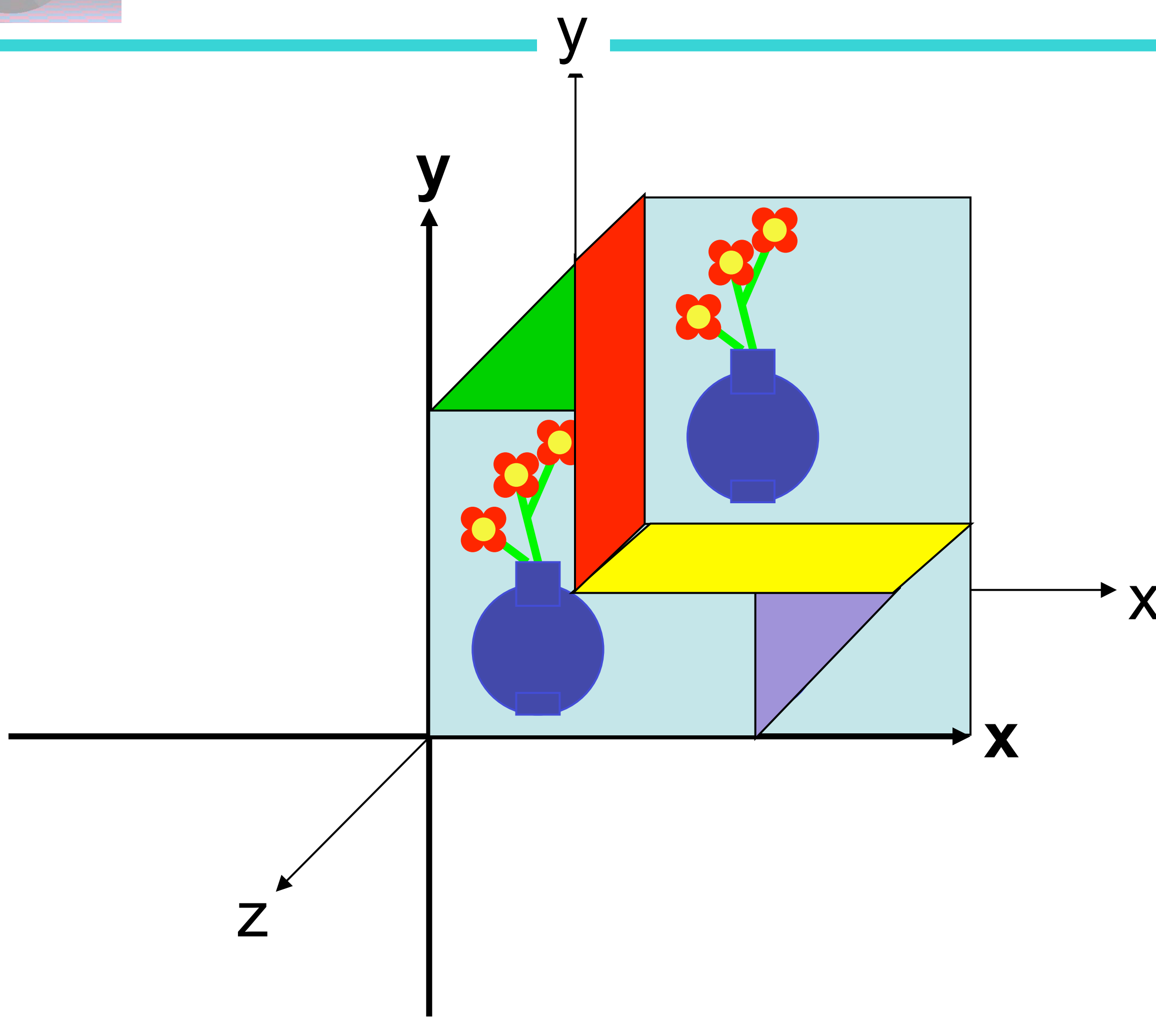
# 2D Linear Translations as 3D Matrices

Any 2D translation can be represented by a 3x3 matrix.

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

This is a 3D shear that acts as a translation on the plane z = 1.

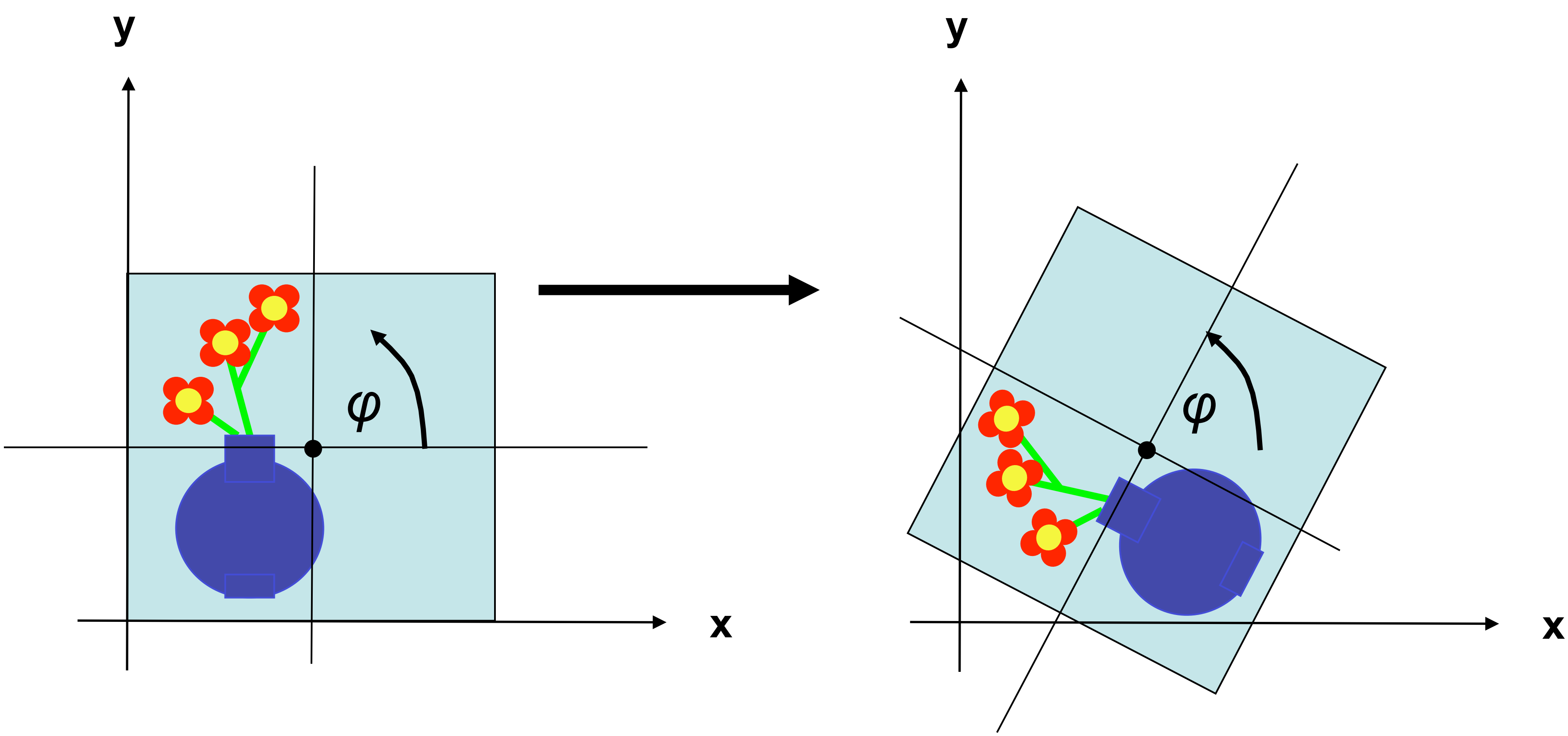
# Translation as a Shear



# 2D Affine Transformations

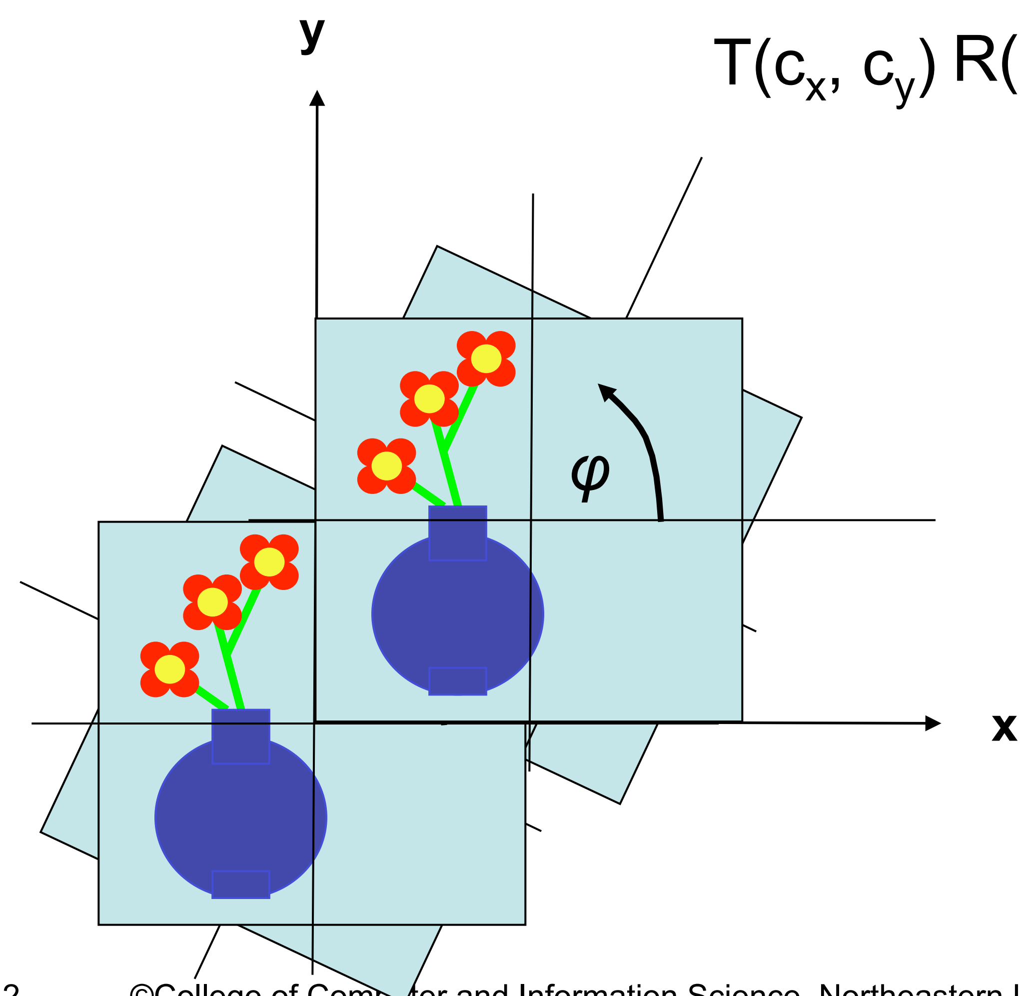
- An *affine transformation* is any transformation that preserves co-linearity (i.e., all points lying on a line initially still lie on a line after transformation) and ratios of distances (e.g., the midpoint of a line segment remains the midpoint after transformation).
- With homogeneous coordinates, we can represent all 2D affine transformations as 3D linear transformations.
- We can then use matrix multiplication to transform objects.

# Rotation about an Arbitrary Point

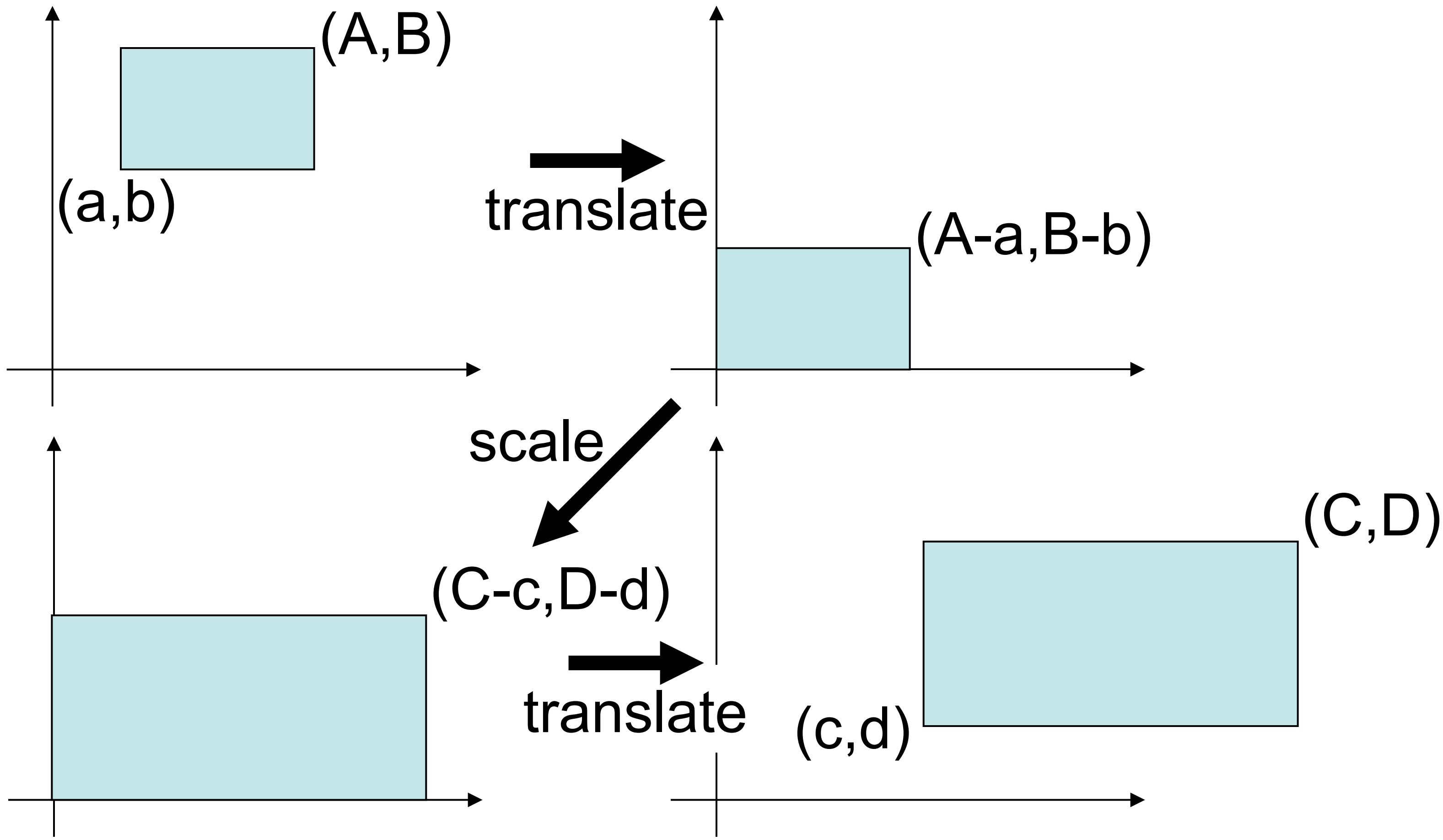


# Rotation about an Arbitrary Point

$T(c_x, c_y)\, R(\varphi)\, T(-c_x, -c_y)$



# Windowing Transforms



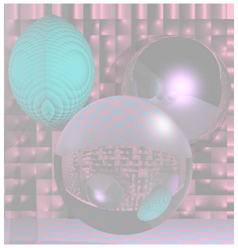
# 3D Transformations

Remember:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \leftrightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

A 3D linear transformation can be represented by a 3x3 matrix.

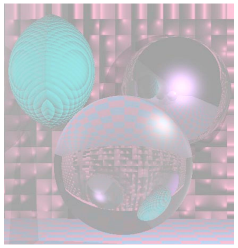
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \leftrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



# 3D Affine Transformations

$$\text{scale}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{translate}(t_x, t_y, t_z) = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



# 3D Rotations

$$\text{rotate}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rotate}_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{rotate}_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$