Written Homework 04

Assigned: Wed 1 Apr 2009
Due: Thu 9 Apr 2009

Instructions:

- The assignment is due at the *beginning* of class on the due date specified. Late assignments will be penalized 50%, as stated in the course information sheet. Late assignments *will not be accepted* after the solutions have been distributed.

- We expect that you will study with friends and often work out problem solutions together, but *you must write up your own solutions, in your own words*. Cheating will not be tolerated. The instructor and the TA will be available to answer questions but will not do your homework for you. One of the course goals is to teach you how to think on your own.

- We expect your homework to be neat, organized, legible, and *stapled*. If your handwriting is unreadable, please type. We will *not* accept pages that are ripped from a spiral notebook. Please use 8.5in by 11in loose-leaf or printer paper.

Problem 1 [15 pts (5 each)]**: Sums**

i. Evaluate the following sum. You must show your work, and your final answer should be a single integer.

$$\sum_{k=5}^{13} 3k =$$

ii. Derive a formula in terms of n for the following sum. You must show your work, and your final formula should only contain n and integers (but not k).

$$\sum_{k=1}^{n} 3k =$$

iii. Derive a formula in terms of a and b for the following sum. You must show your work, and your final formula should only contain a, b, and integers (but not k).

$$\sum_{k=5}^{13} a \cdot b^k =$$

Problem 2 [30 pts (15 each)]: **Recurrences**

Solve the following two recurrences using the method described in class and in the text. *You must show your work, otherwise you will lose points.*

Assume a base case of $T(1) = 1$. As part of your solution, you will need to establish a pattern for what the recurrence looks like after the k-th iteration. You *do not* need to formally prove that your patterns are correct via induction, but you will lose points if your patterns are not correct. Your solutions may include integers, n raised to a power, and/or logarithms of n. For example, a solution of the form $8^{\log_2 n}$ is unacceptable; this should be simplified as $n^{\log_2 8} = n^3$.

   i. $T(n) = 4\,T(n/2) + n^2$.

   ii. $T(n) = 25\,T(n/5) + n$.

Problem 3 [15 pts (5,10)]: **Binary Search**

We saw the binary search algorithm both in class and in the text. In this problem, you will develop a formula $T(n)$ for the worst case number of comparisons made by the binary search algorithm.

   i. When performing binary search on an array of size n, we make two comparisons (a "=" followed by a ">") to determine which half of the array the element lies in, and then we recursively search the correct half. Give the recurrence formula which captures this algorithm. Your answer should look like $T(n) =$_____, where the right hand side contains a recursive call to $T()$ plus any additional values necessary.

   ii. Solve the recurrence from part **i**, using the base case $T(1) = 1$. Point out any differences between the answer you got here and the one we got in class, and explain why the differences are or are not important.

Problem 4 [40 pts (5 each)]: **Graphs**

A (simple) graph consists of a set of vertices and a set of (undirected) edges, where each edge connects a pair of vertices. Let $V_3 = \{a, b, c\}$ be a set of three vertices labeled a, b, and c. Now consider all (simple) graphs which can be formed from these three vertices; one obtains different graphs by having different sets of edges.

   i. Draw all possible graphs that can be constructed from the vertices $V_3 = \{a, b, c\}$.

   ii. How many such graphs have no edges? How many such graphs have exactly one edge? How many such graphs have exactly two edges? How many such graphs have all three edges? How many total graphs are there?

Now consider a vertex set of size 4, $V_4 = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$. *You do not need to draw the graphs for these questions.*

iii. How many possible edges exist over V_4?

iv. How many unique graphs can be constructed from V_4? *Hint:* In any such graph, each edge is either present or absent.

v. How many unique graphs can be constructed from V_4 with exactly three edges? *Hint:* One must *choose* where to place the three edges...

Now generalize these results for a vertex set of size n, i.e., $V_n = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots\}$ where $|V_n| = n$.

vi. How many possible edges exist over V_n? Justify your answer.

vii. How many unique graphs can be constructed from V_n? Justify your answer.

viii. How many unique graphs can be constructed from V_n using exactly k edges, where k is some number between 0 and the answer to part **vi**? Justify your answer.