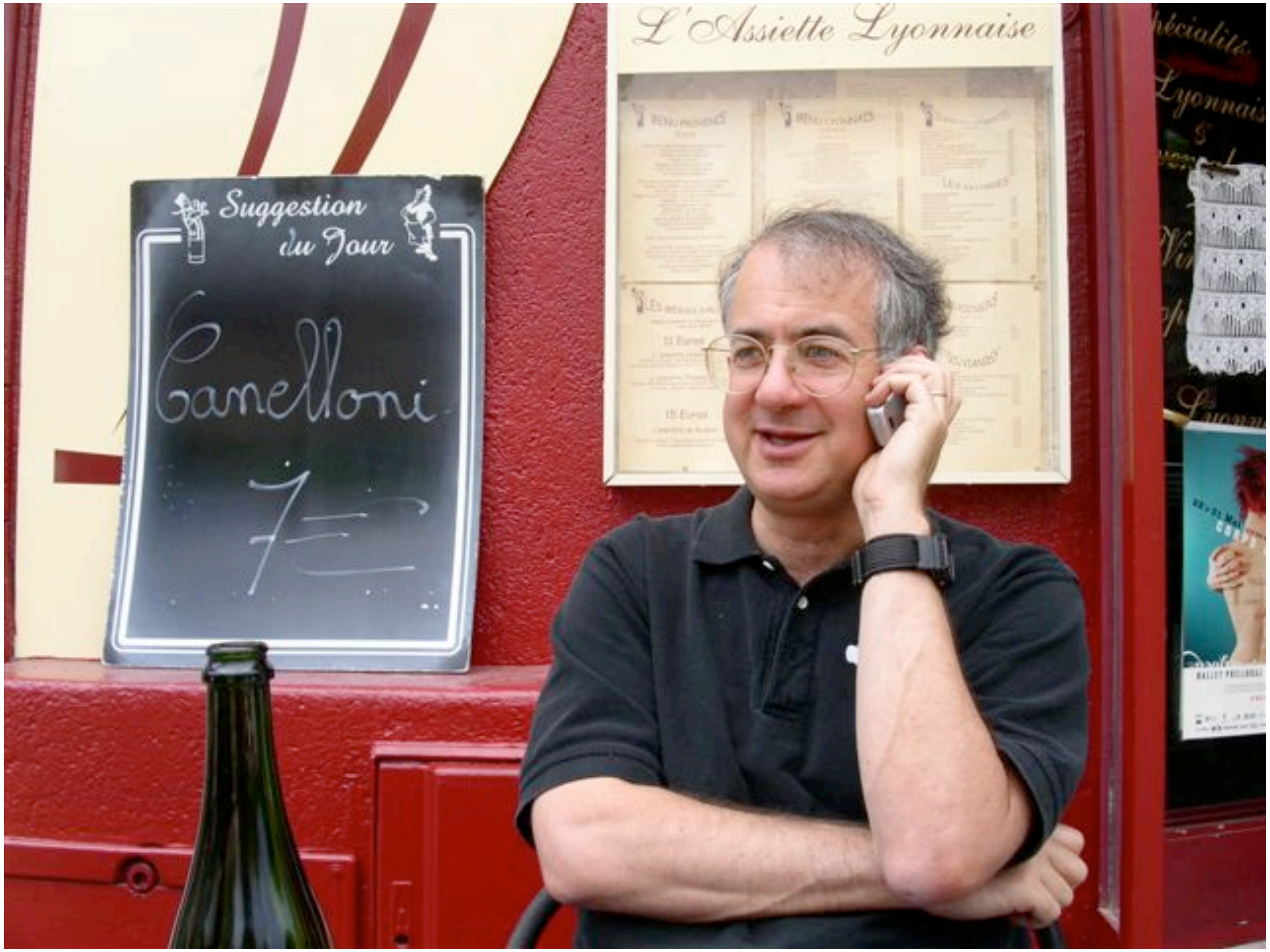
The Complexity of *k*CFA

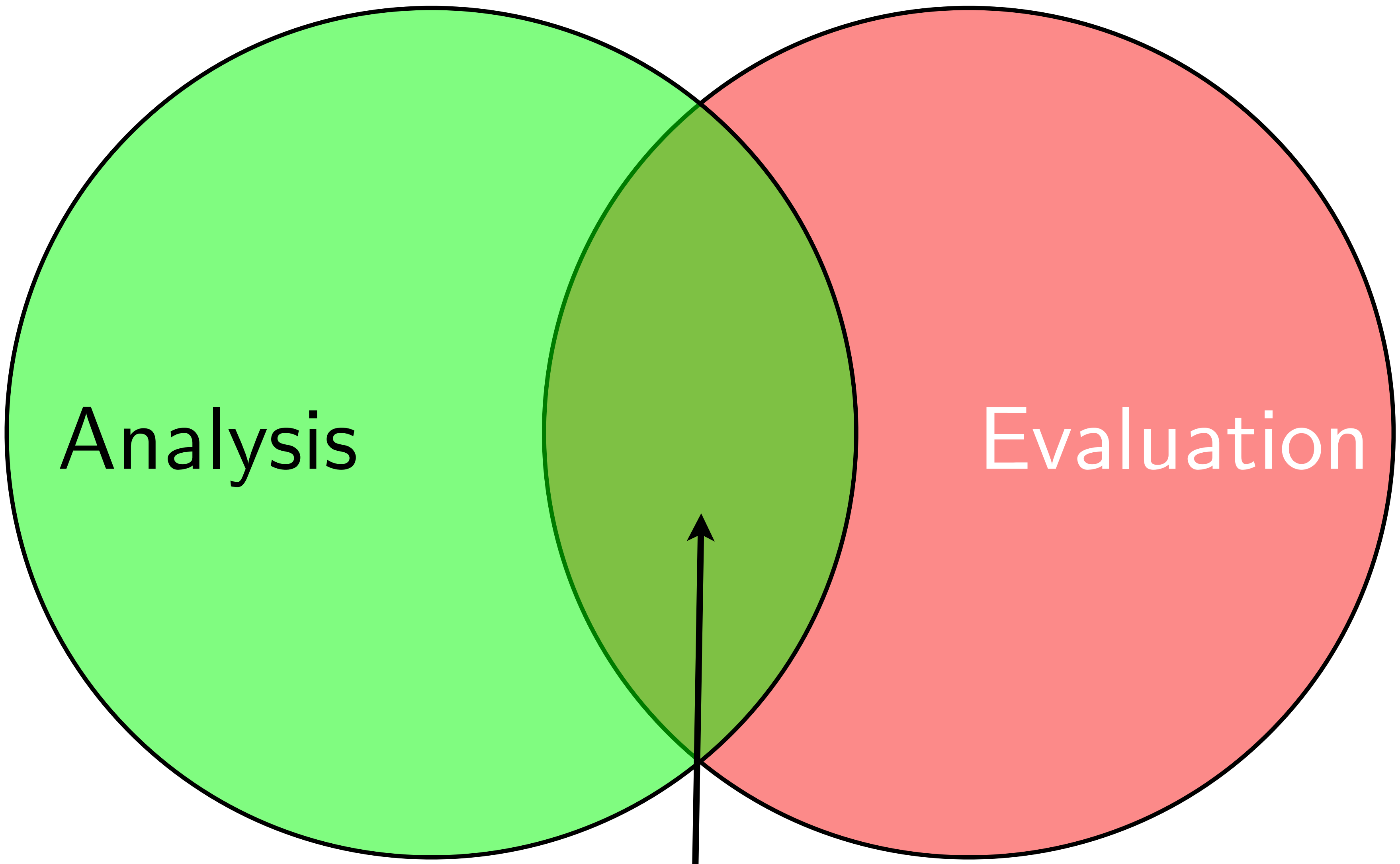
David Van Horn

& Harry Mairson

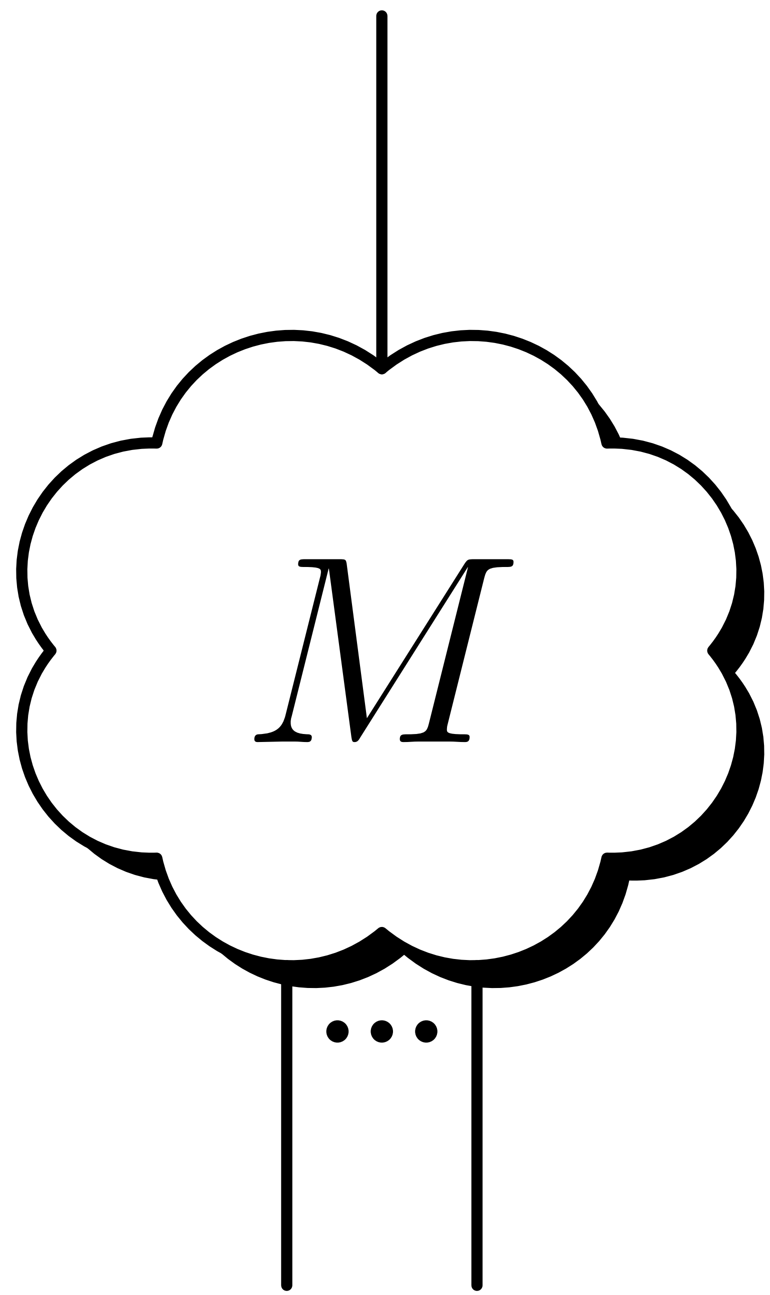
Van Horn and Mairson, ICFP'07, ICFP'08, SAS'08



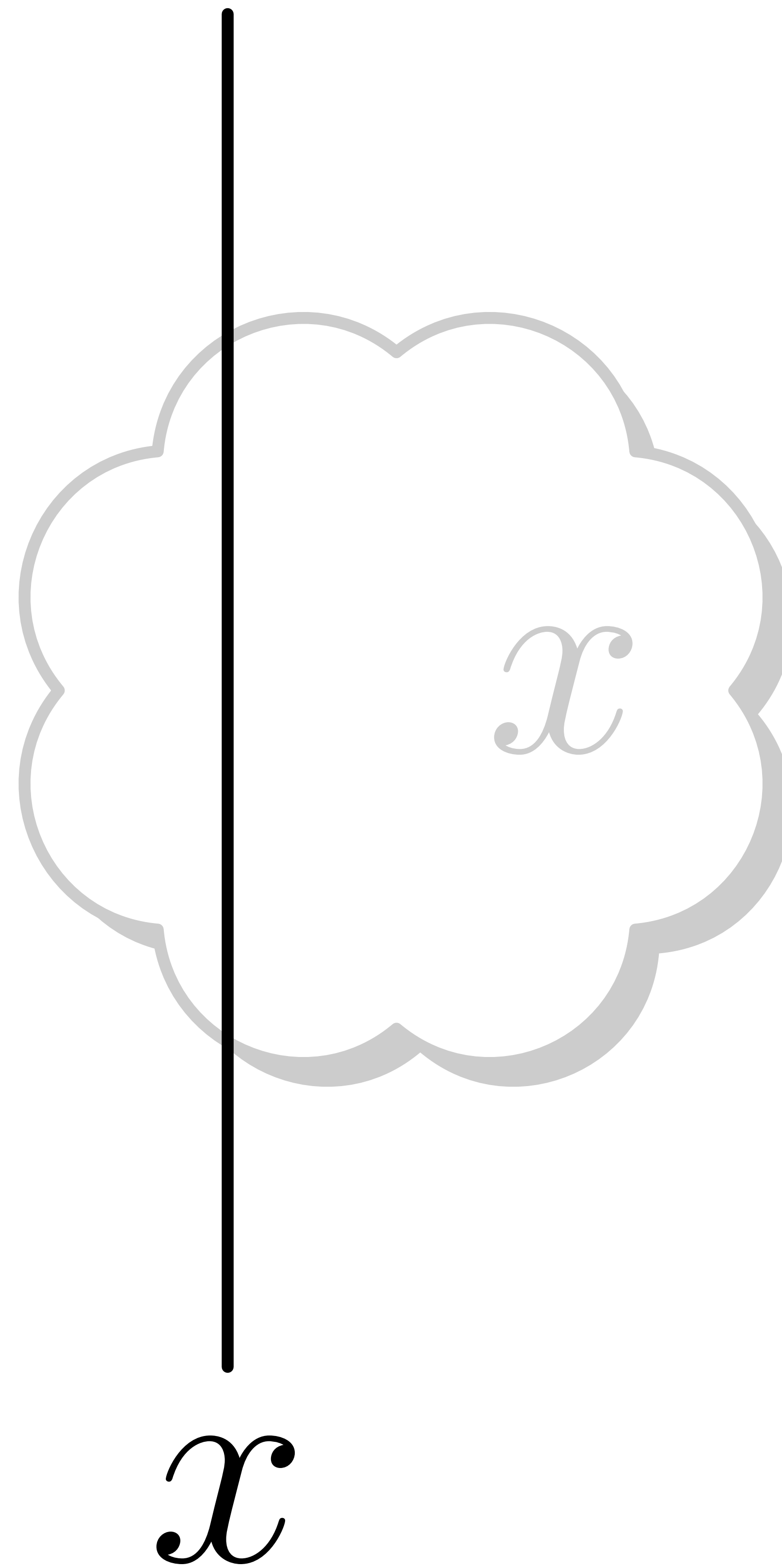
0CFA and PTIME

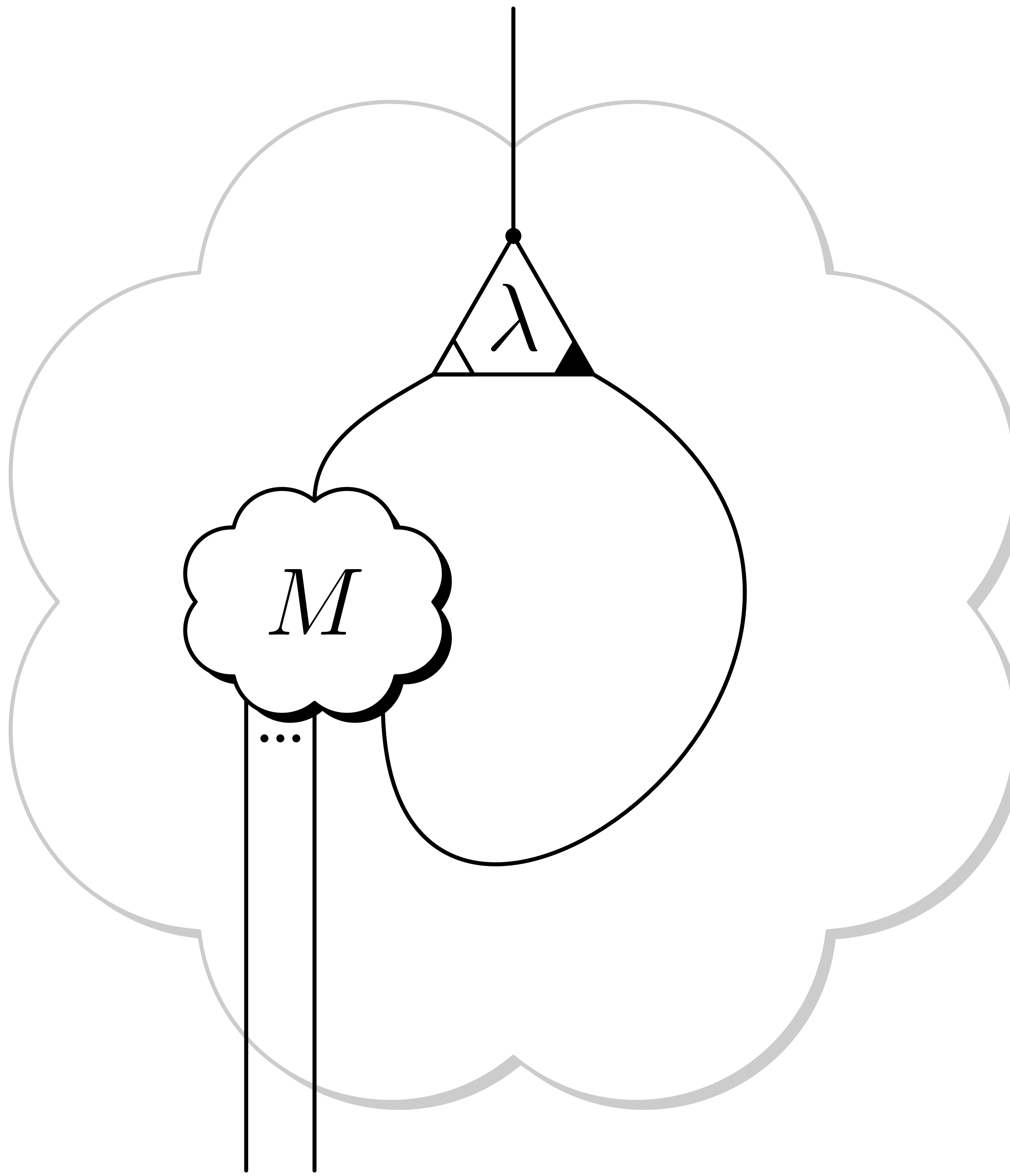


What is in the intersection?

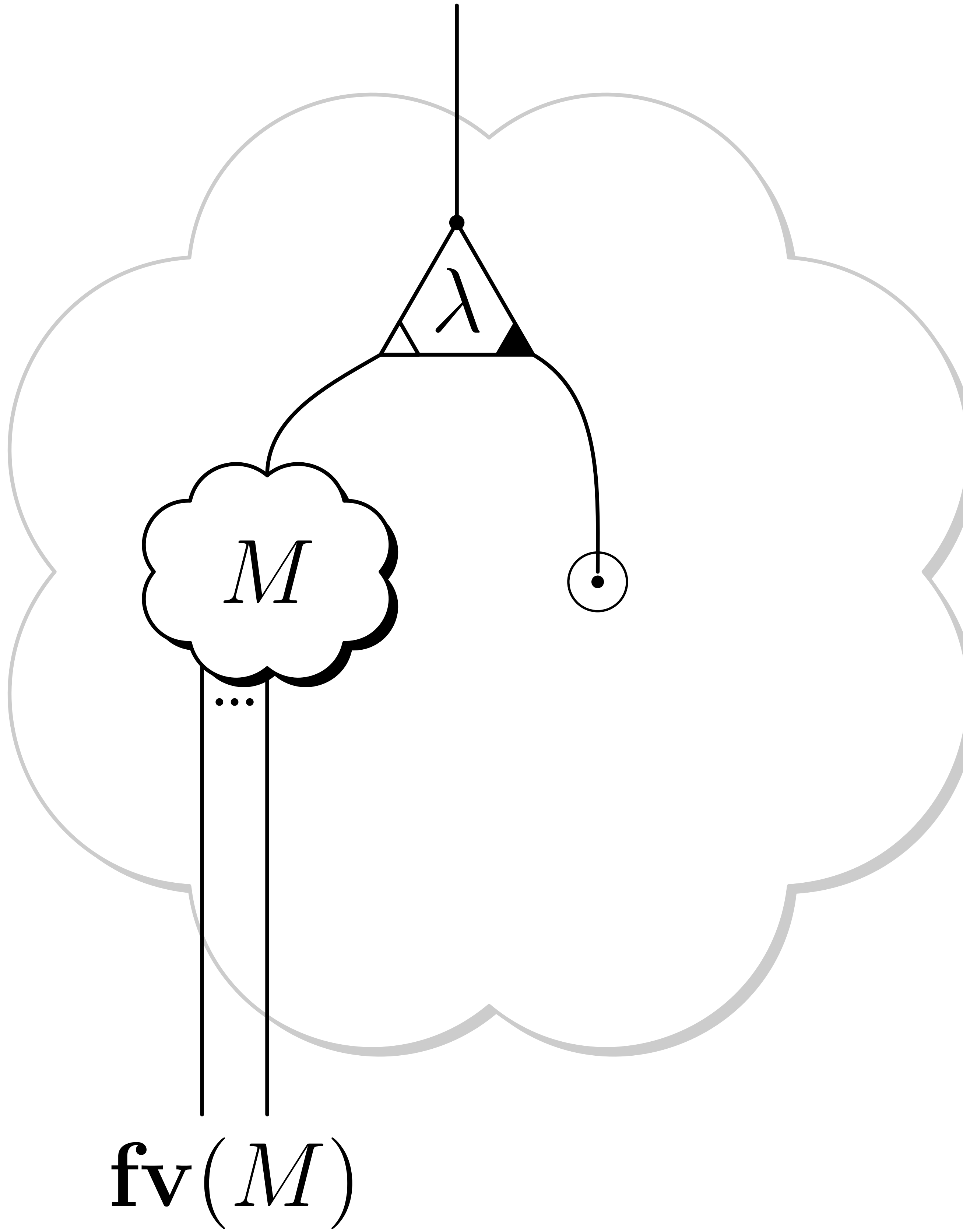


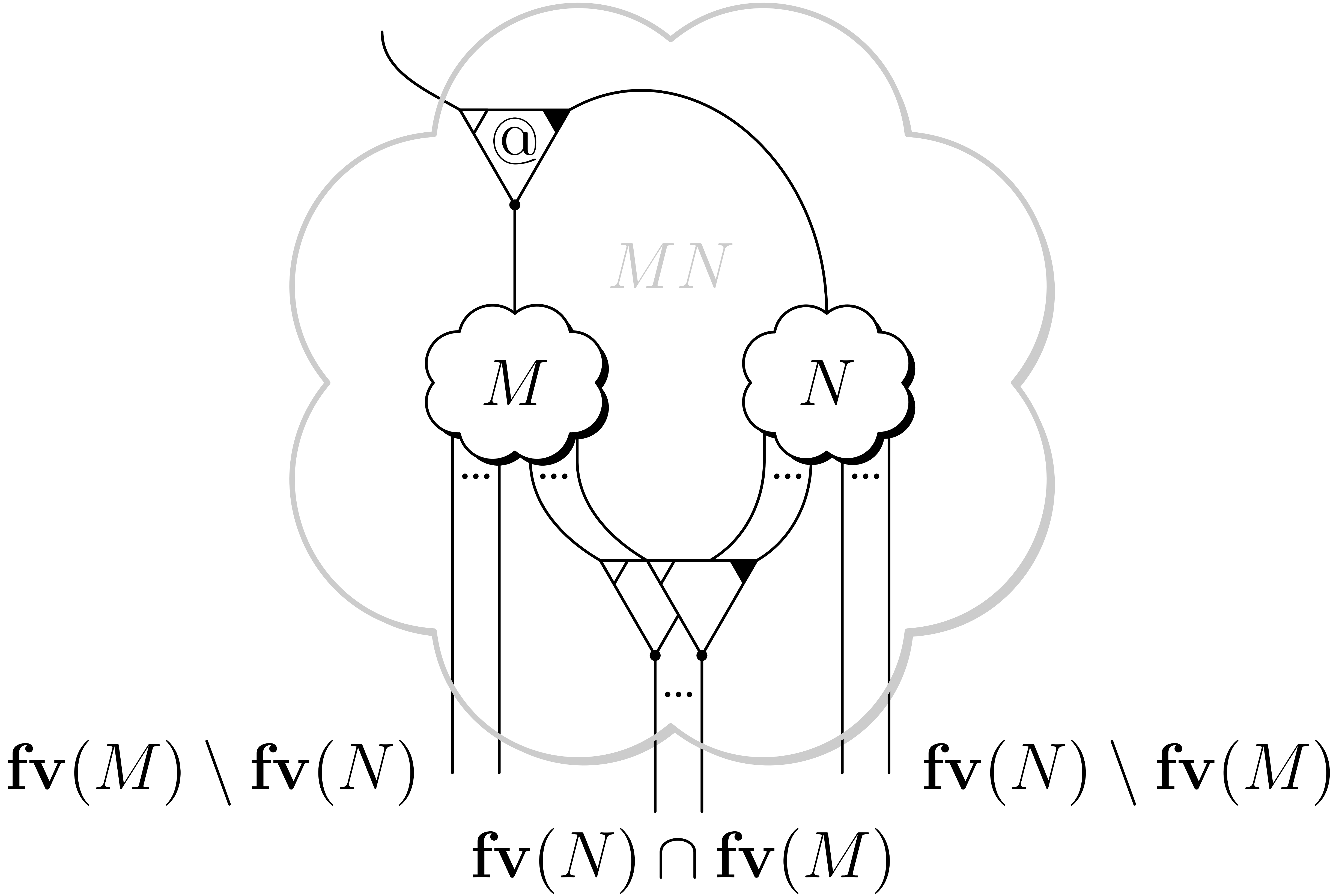
$$\mathbf{fv}(M)$$

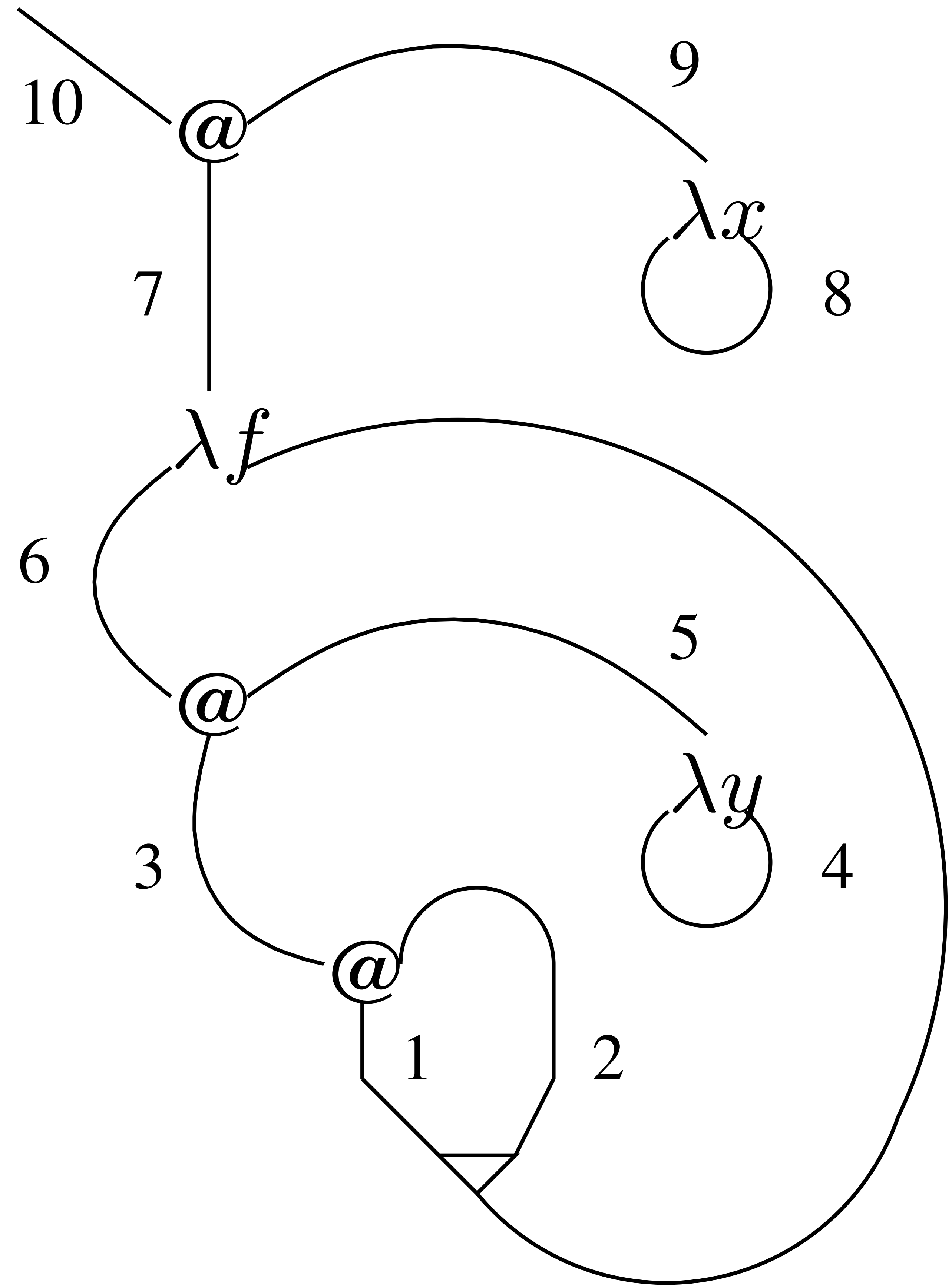

x
x

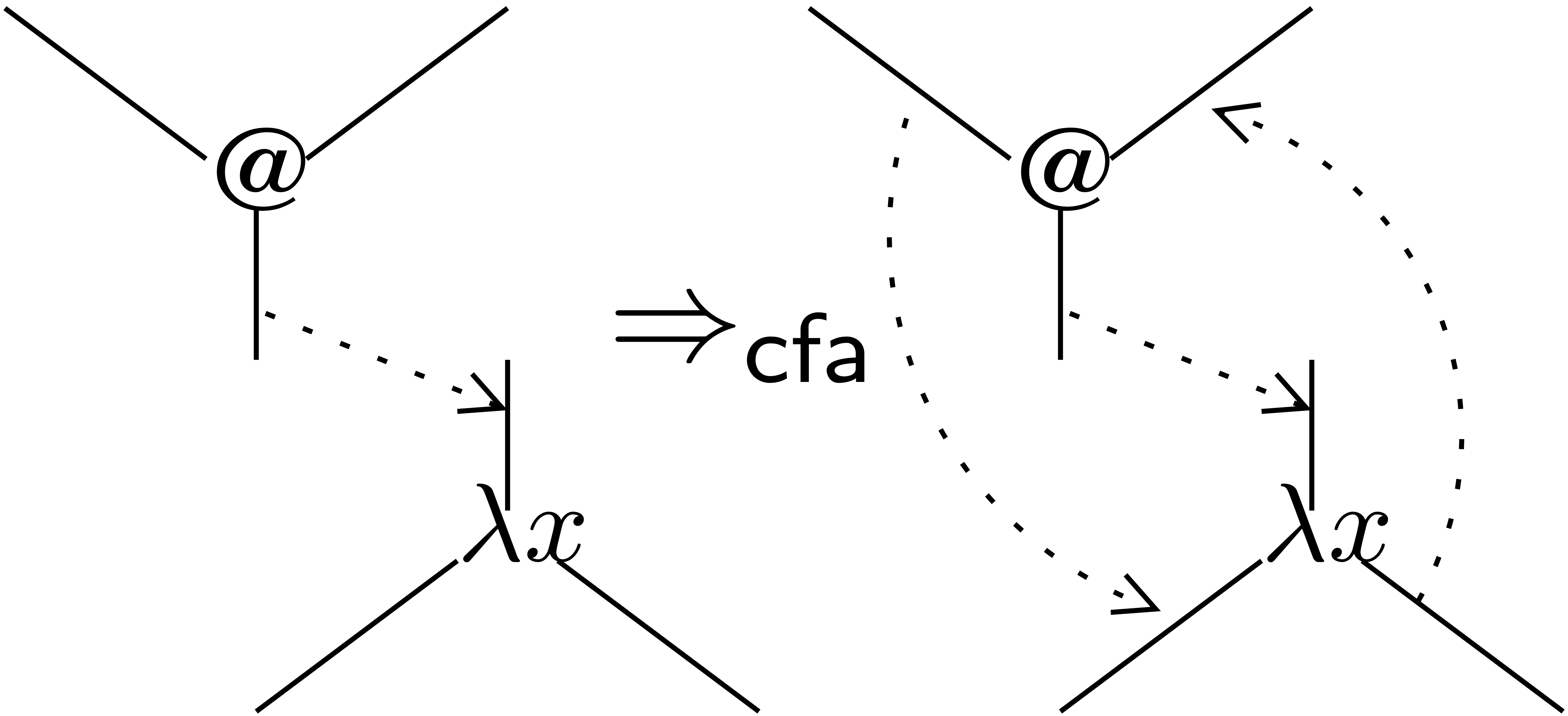


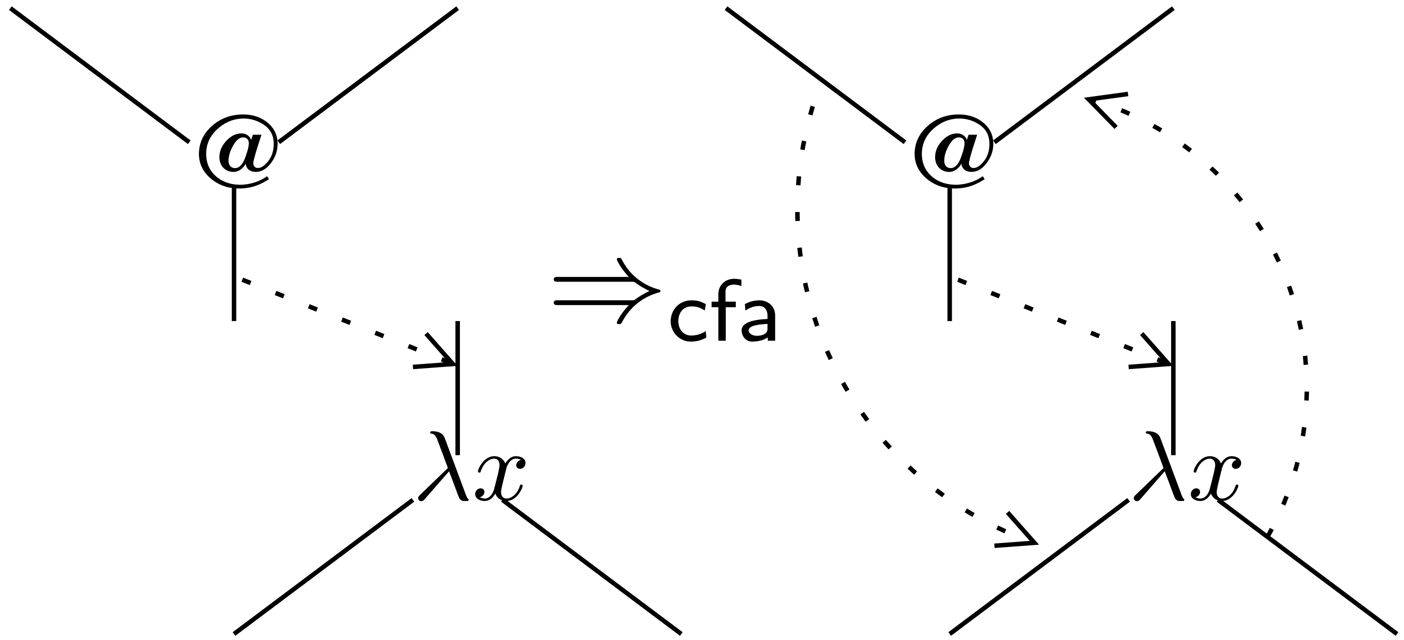
$$\mathbf{fv}(M) \setminus \{x\}$$

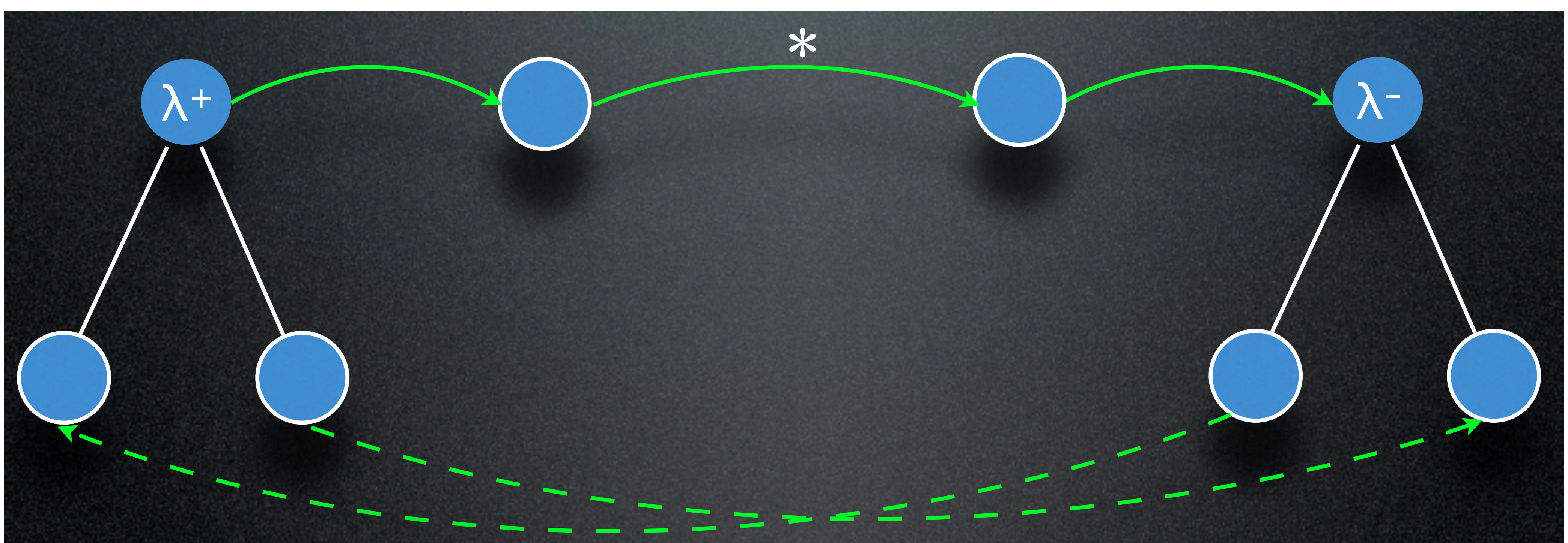

λ
M
$\mathbf{fv}(M)$

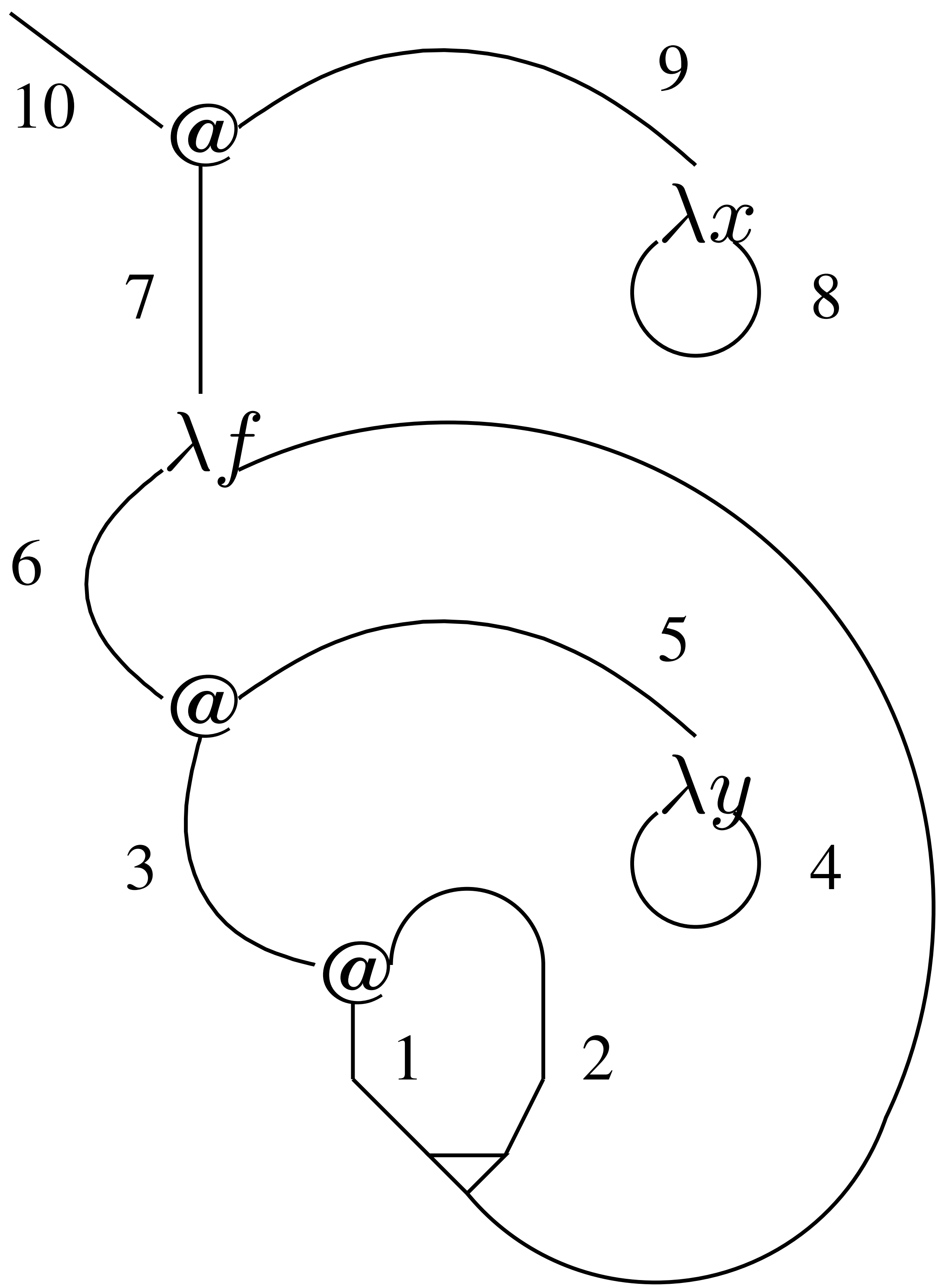

@
MN
M
N
$\mathbf{fv}(M) \setminus \mathbf{fv}(N)$
$\mathbf{fv}(N) \setminus \mathbf{fv}(M)$
$\mathbf{fv}(N) \cap \mathbf{fv}(M)$


10
@
9
λx
7
8
λf
6
5
@
λy
3
4
@
1
2

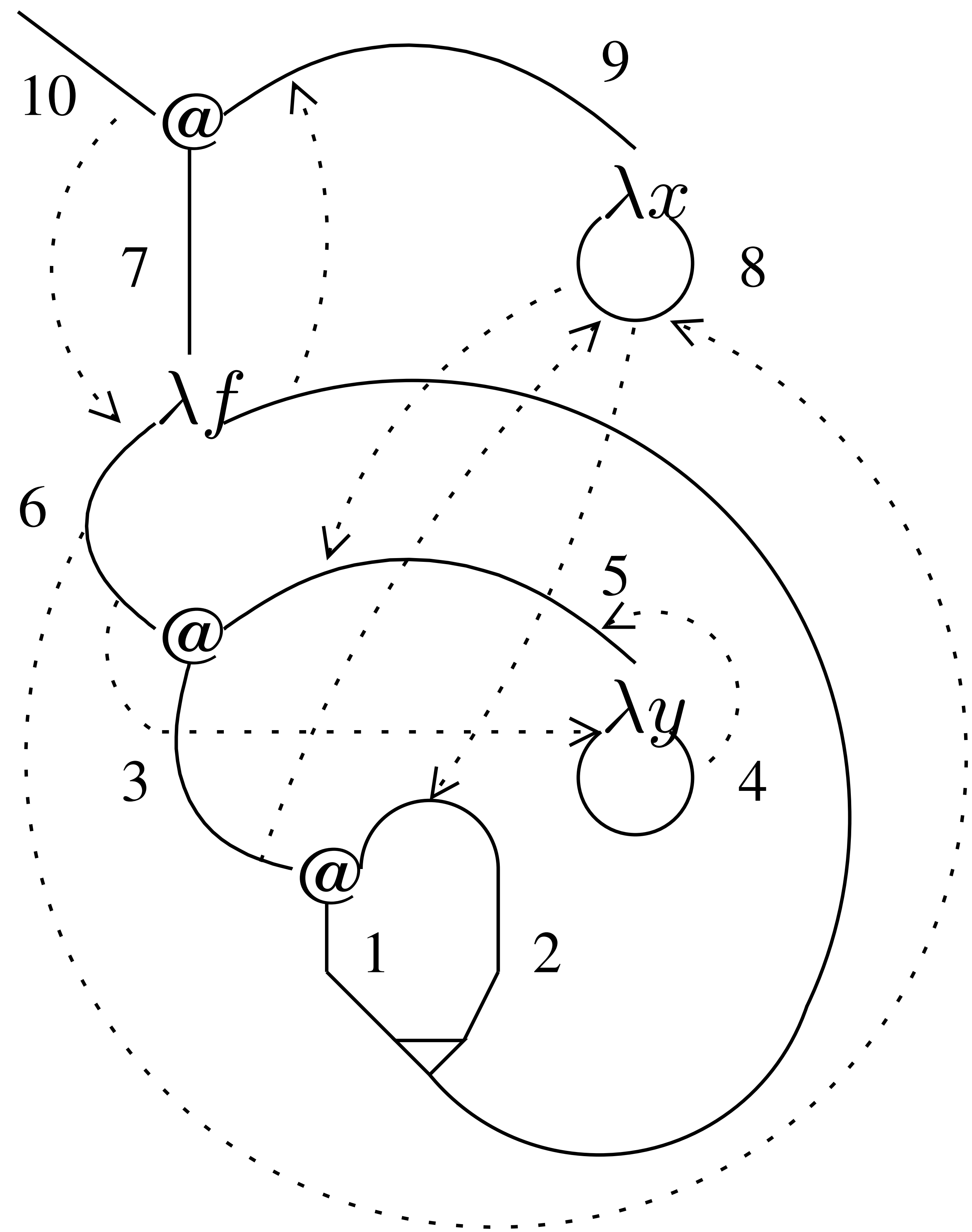

@
λx
⇒cfa
@
λx


@
⇒cfa
@
λx
λx

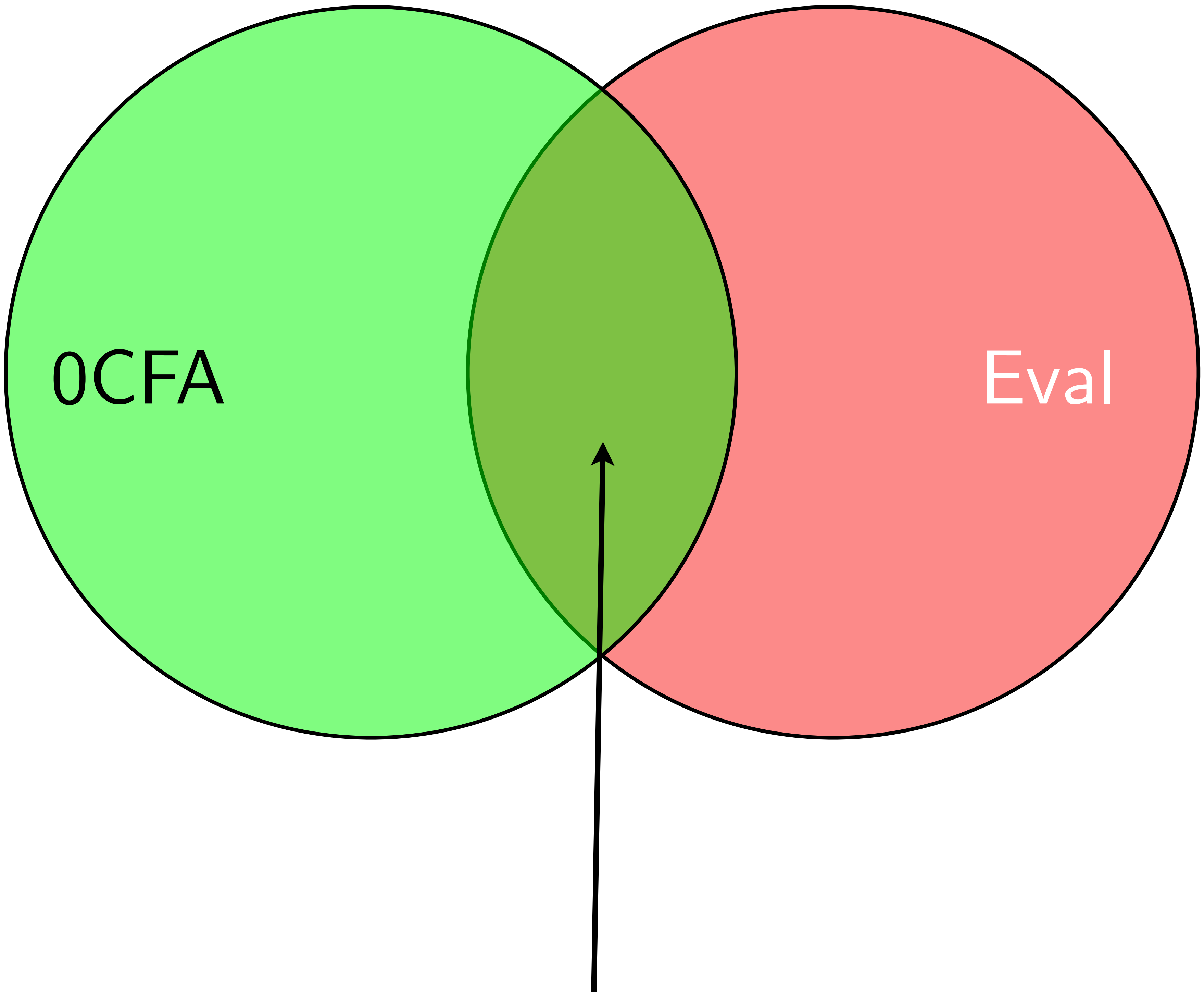

λ+
*
λ−



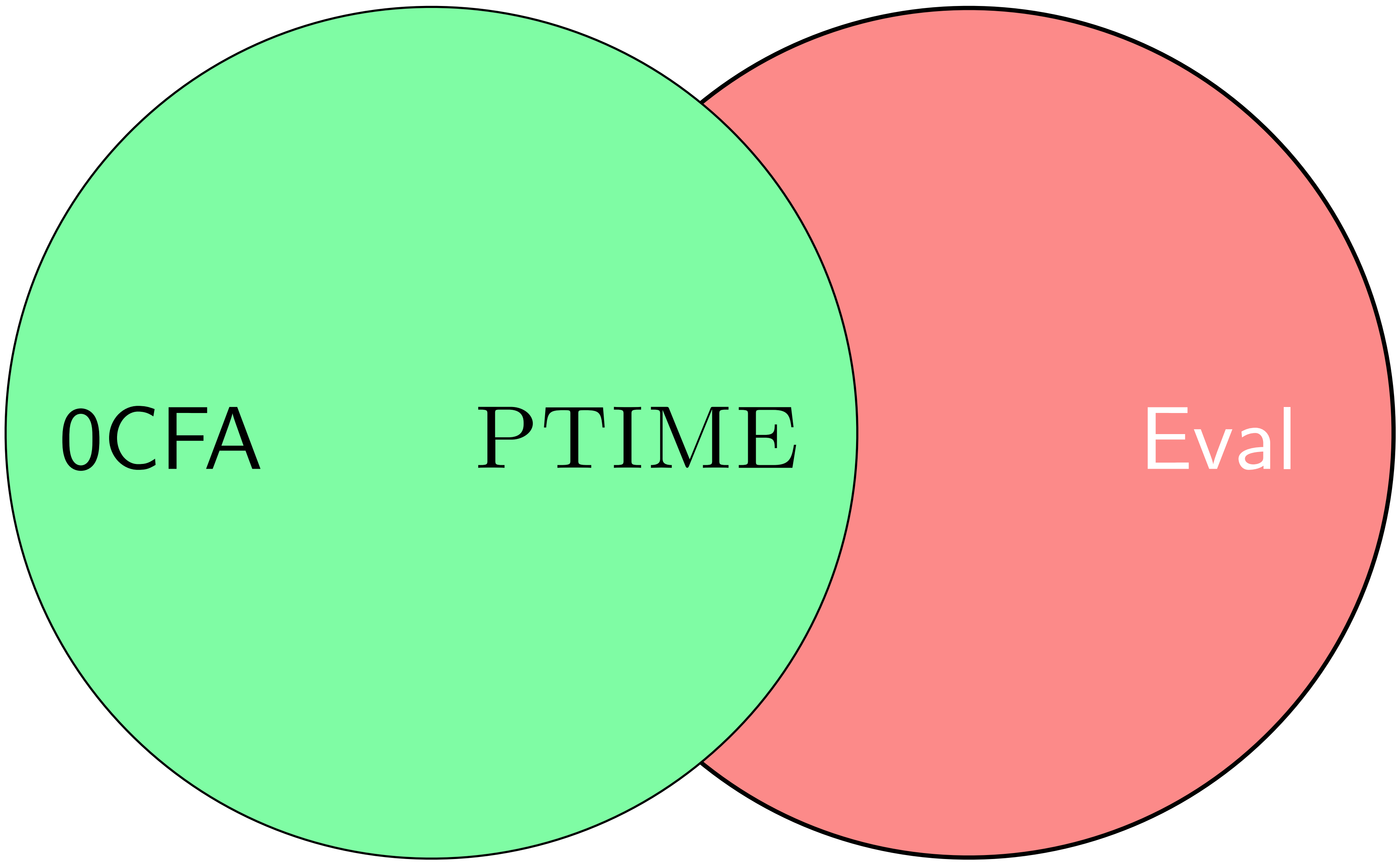
10
@
9
λx
7
8
λf
6
@
5
λy
3
4
@
1
2

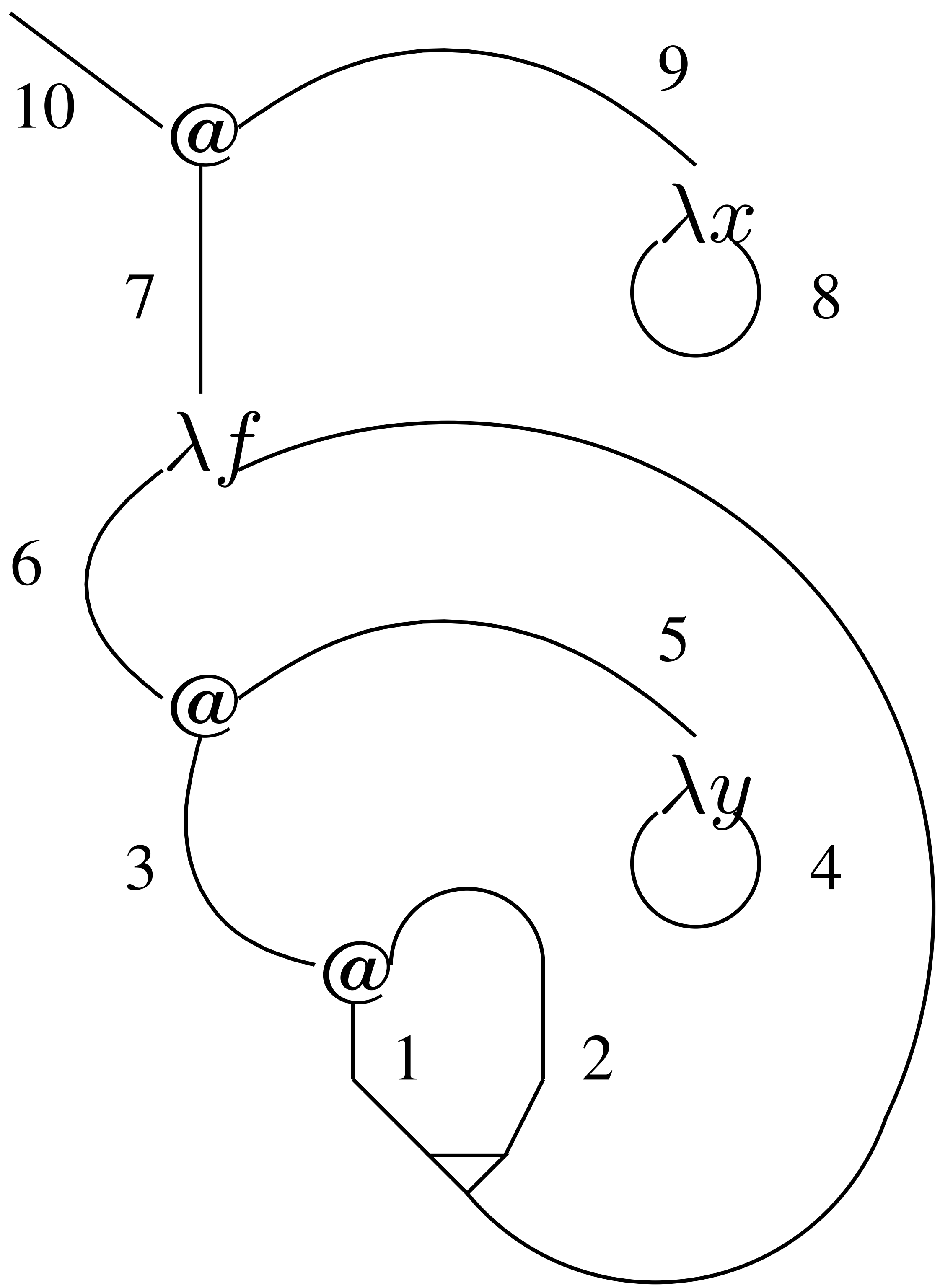


10
@
9
λx
7
8
λf
6
@
5
λy
3
4
@
1
2

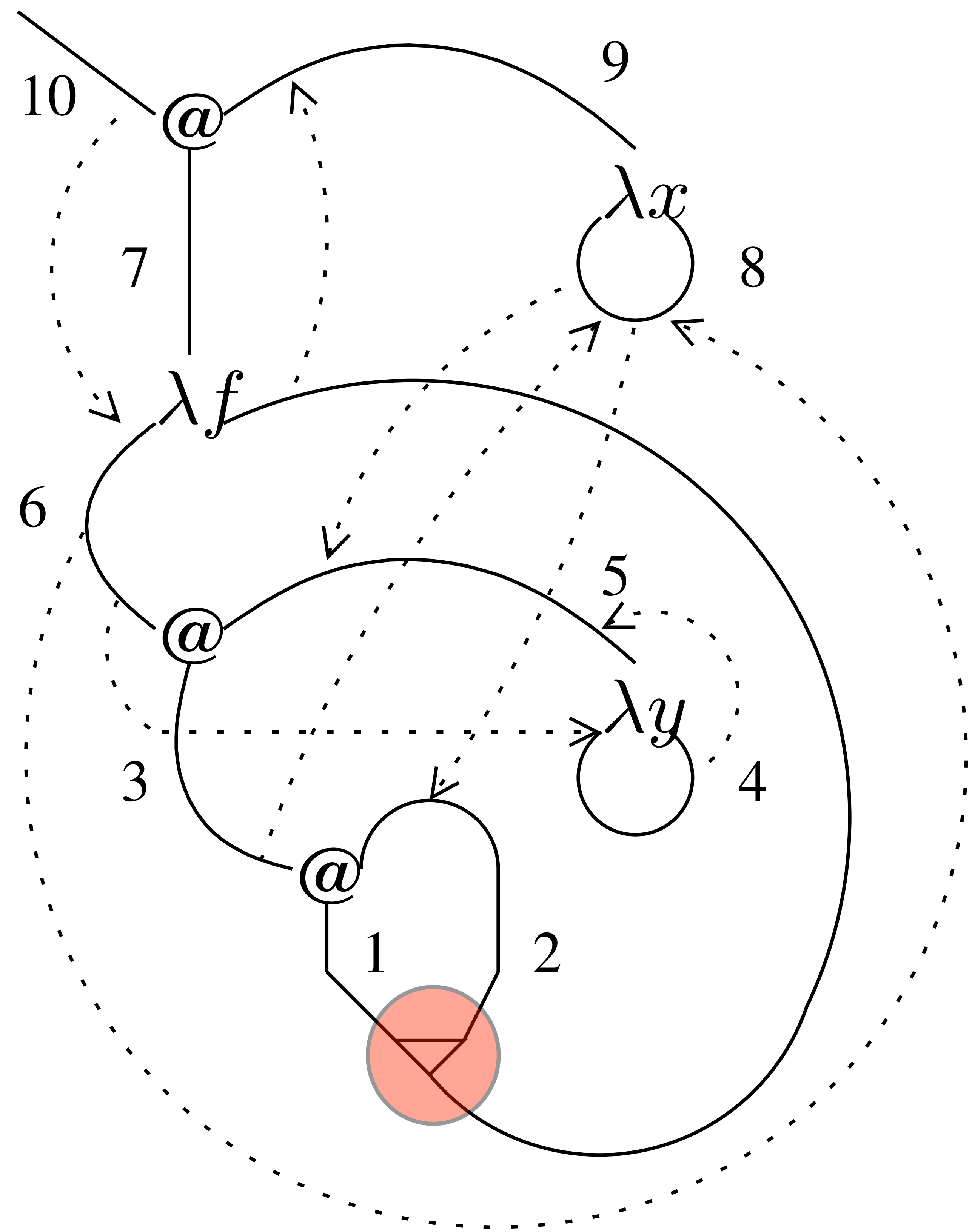


Every variable occurs once.

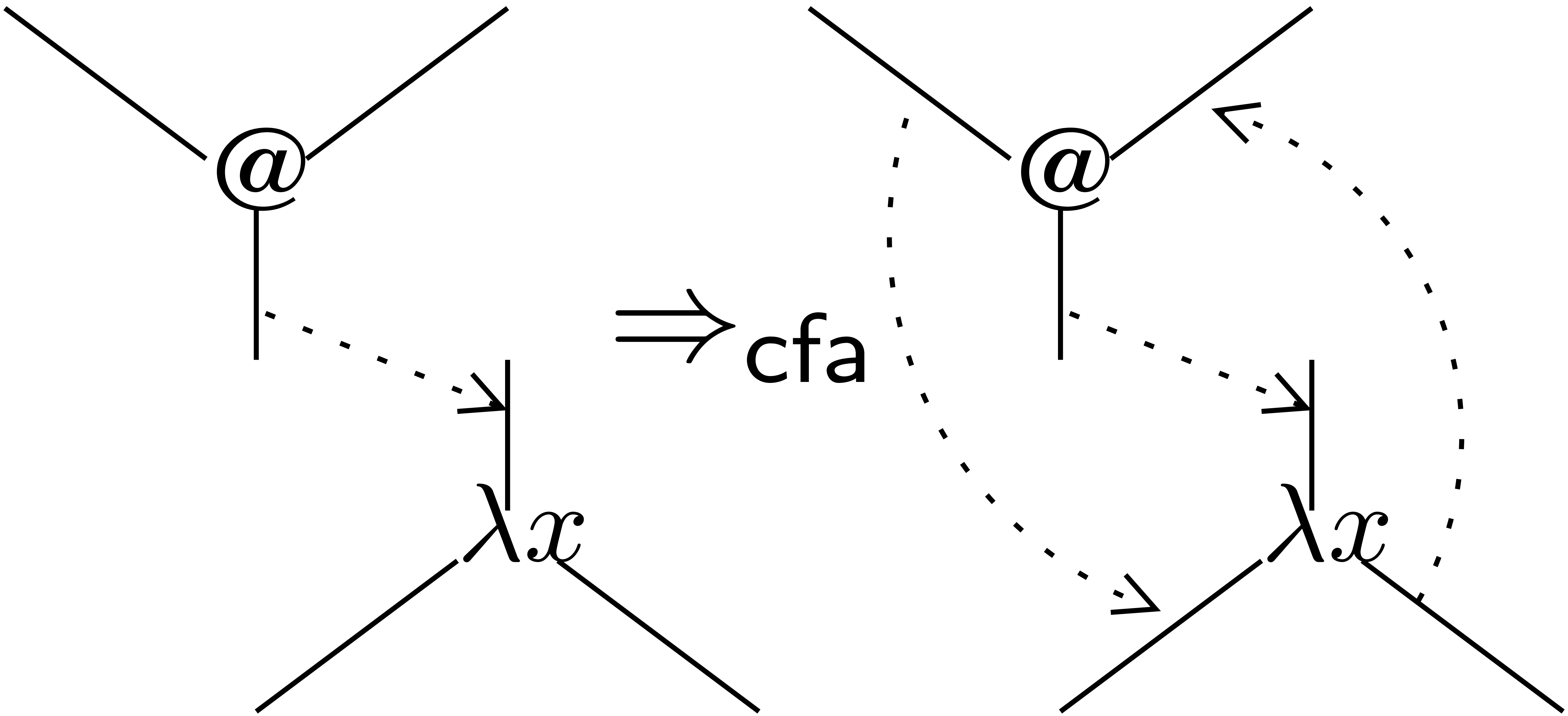

0CFA
PTIME
Eval

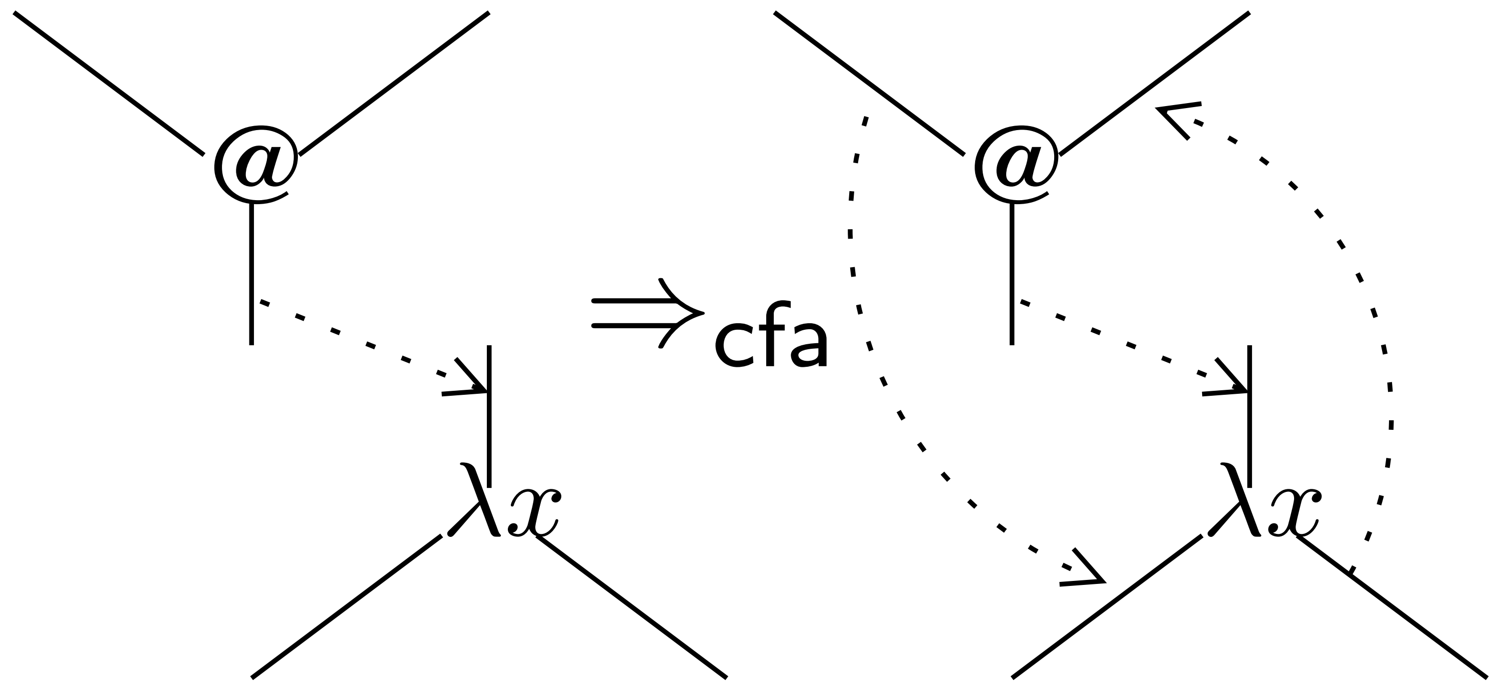


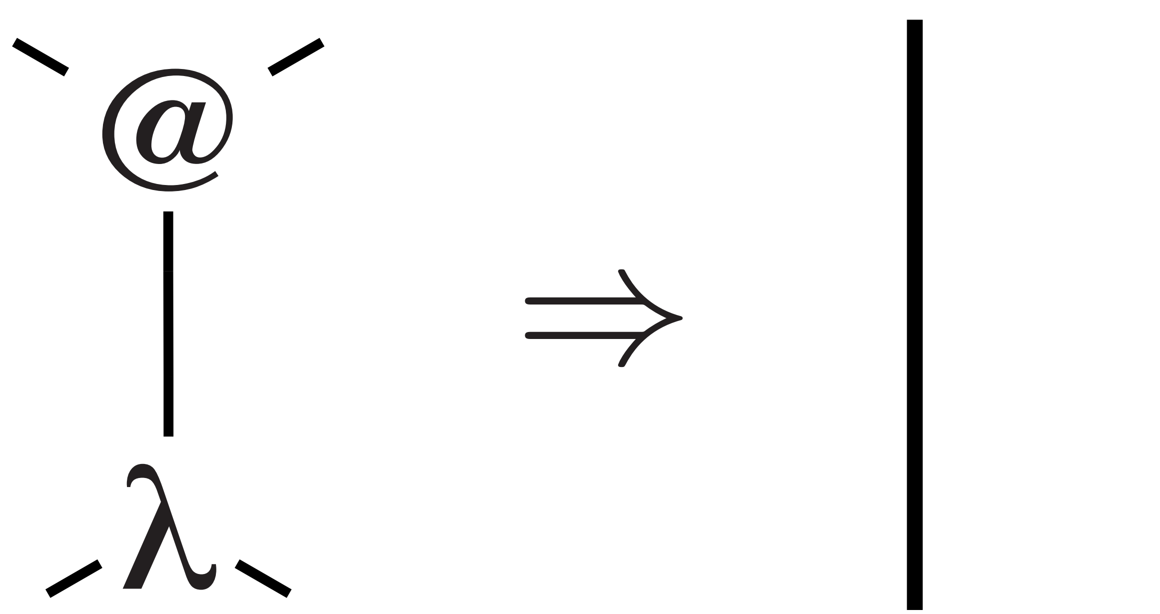
10
9
@
λx
7
8
λf
6
5
@
λy
3
4
@
1
2



10
9
@
λx
7
8
λf
6
5
@
λy
3
4
@
1
2


@
λx
$\Rightarrow_{\mathsf{cfa}}$
@
λx


@
λx
⇒cfa
@
λx

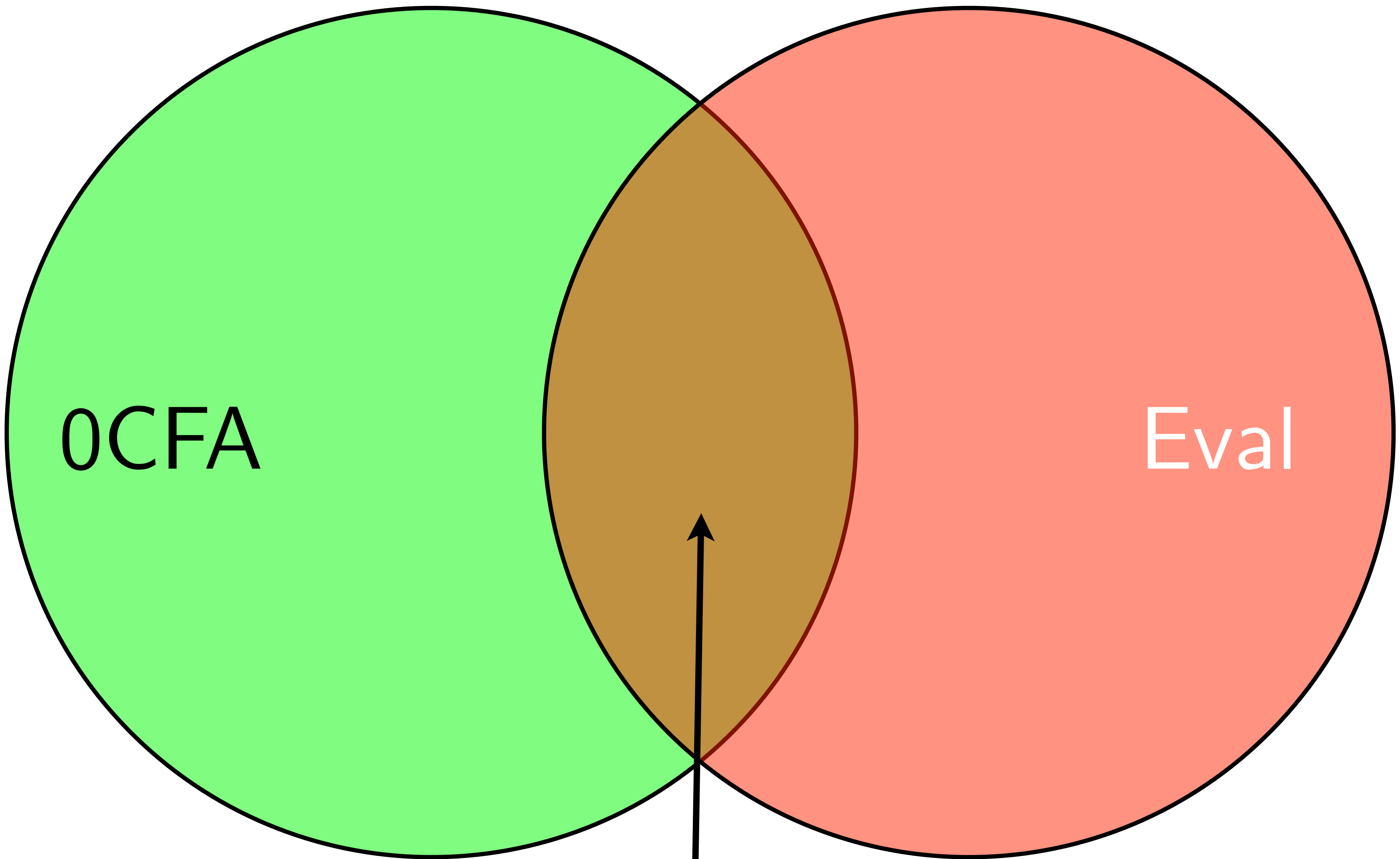

@
λ
⇒

$$\texttt{TT} \equiv \lambda p.\text{let } \langle x, y\rangle = p \text{ in } \langle x, y\rangle \qquad \texttt{True} \equiv \langle \texttt{TT}, \texttt{FF}\rangle$$

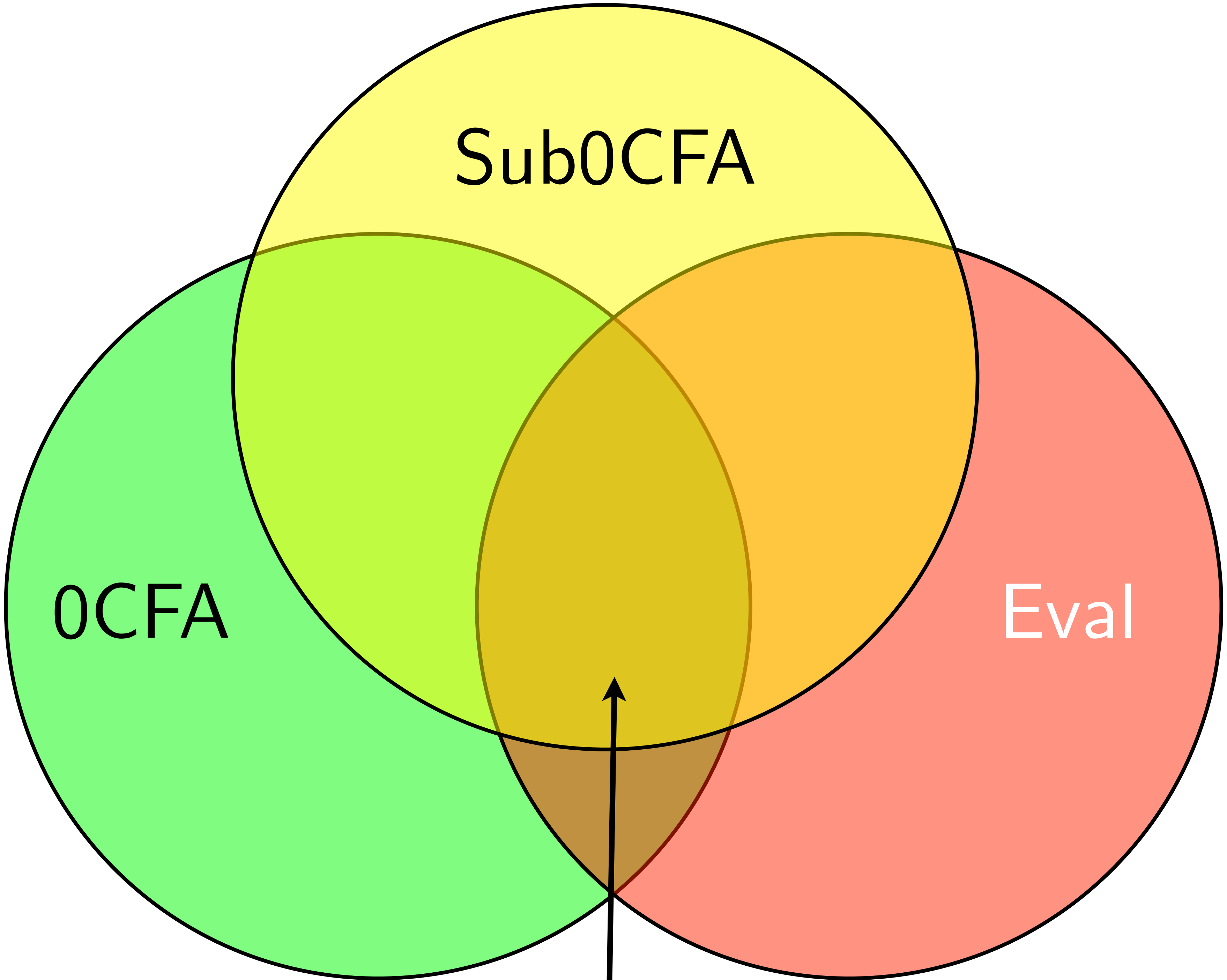
$$\texttt{FF} \equiv \lambda p.\text{let } \langle x, y\rangle = p \text{ in } \langle y, x\rangle \qquad \texttt{False} \equiv \langle \texttt{FF}, \texttt{TT}\rangle$$

$$\texttt{Copy} \equiv \lambda b.\text{let } \langle u, v\rangle = b \text{ in } \langle u\langle \texttt{TT}, \texttt{FF}\rangle, v\langle \texttt{FF}, \texttt{TT}\rangle\rangle$$

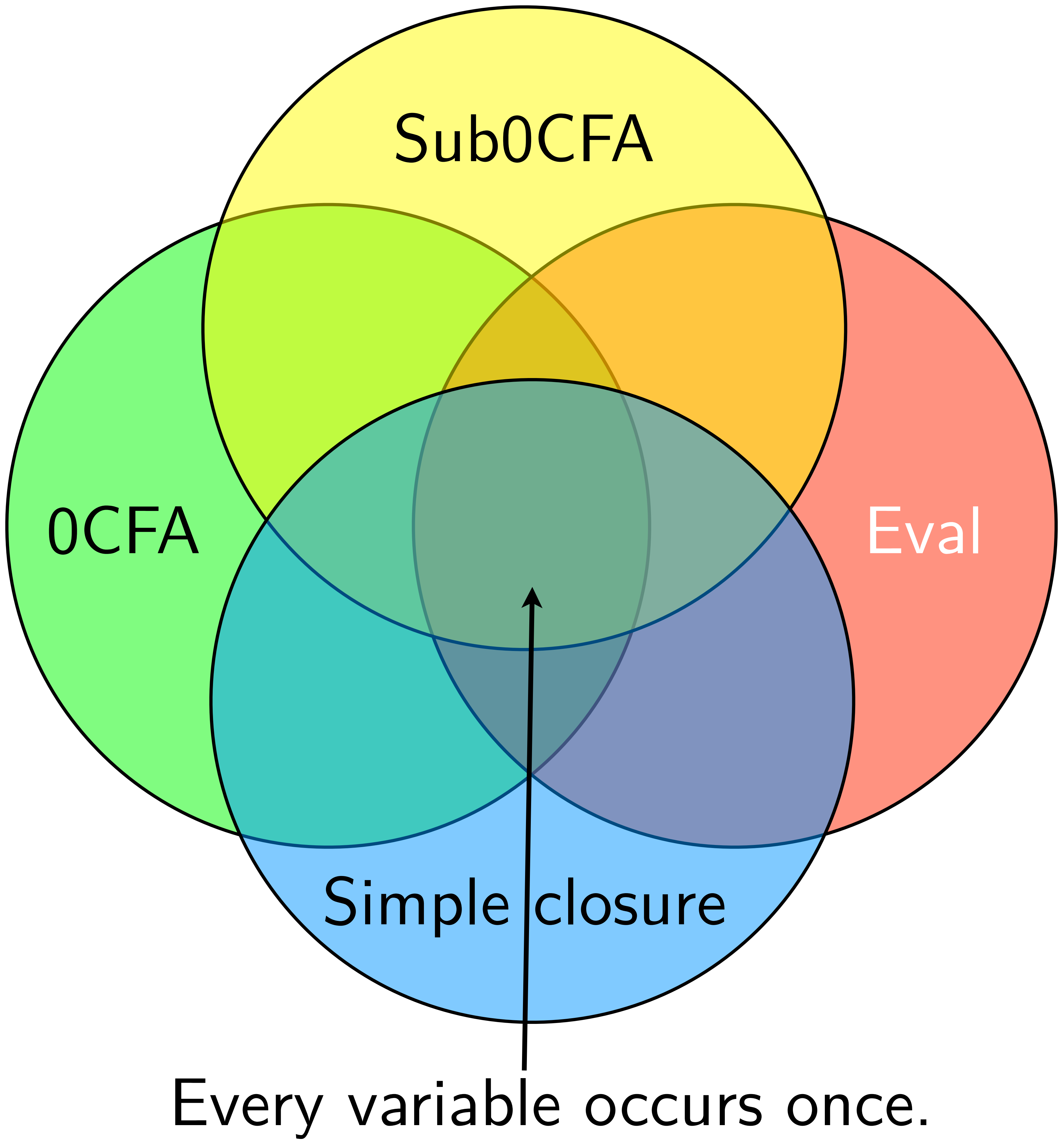
$$\begin{aligned}\texttt{Implies} \equiv\ & \lambda b_1.\lambda b_2. \\ & \text{let } \langle u_1, v_1\rangle = b_1 \text{ in} \\ & \text{let } \langle u_2, v_2\rangle = b_2 \text{ in} \\ & \text{let } \langle p_1, p_2\rangle = u_1\langle u_2, \texttt{TT}\rangle \text{ in} \\ & \text{let } \langle q_1, q_2\rangle = v_1\langle \texttt{FF}, v_2\rangle \text{ in} \\ & \quad \langle p_1, q_1 \circ p_2 \circ q_2 \circ \texttt{FF}\rangle\end{aligned}$$

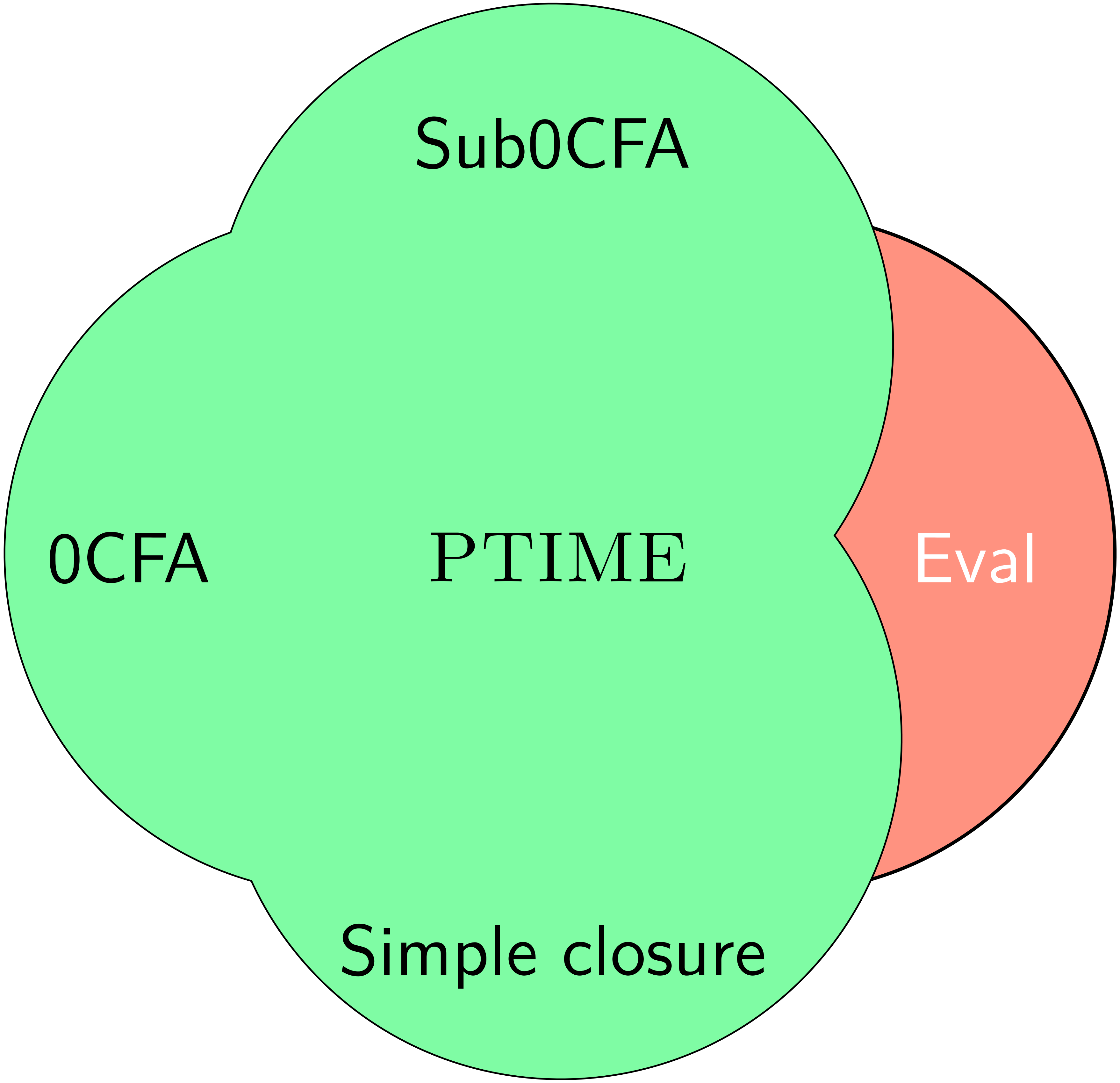


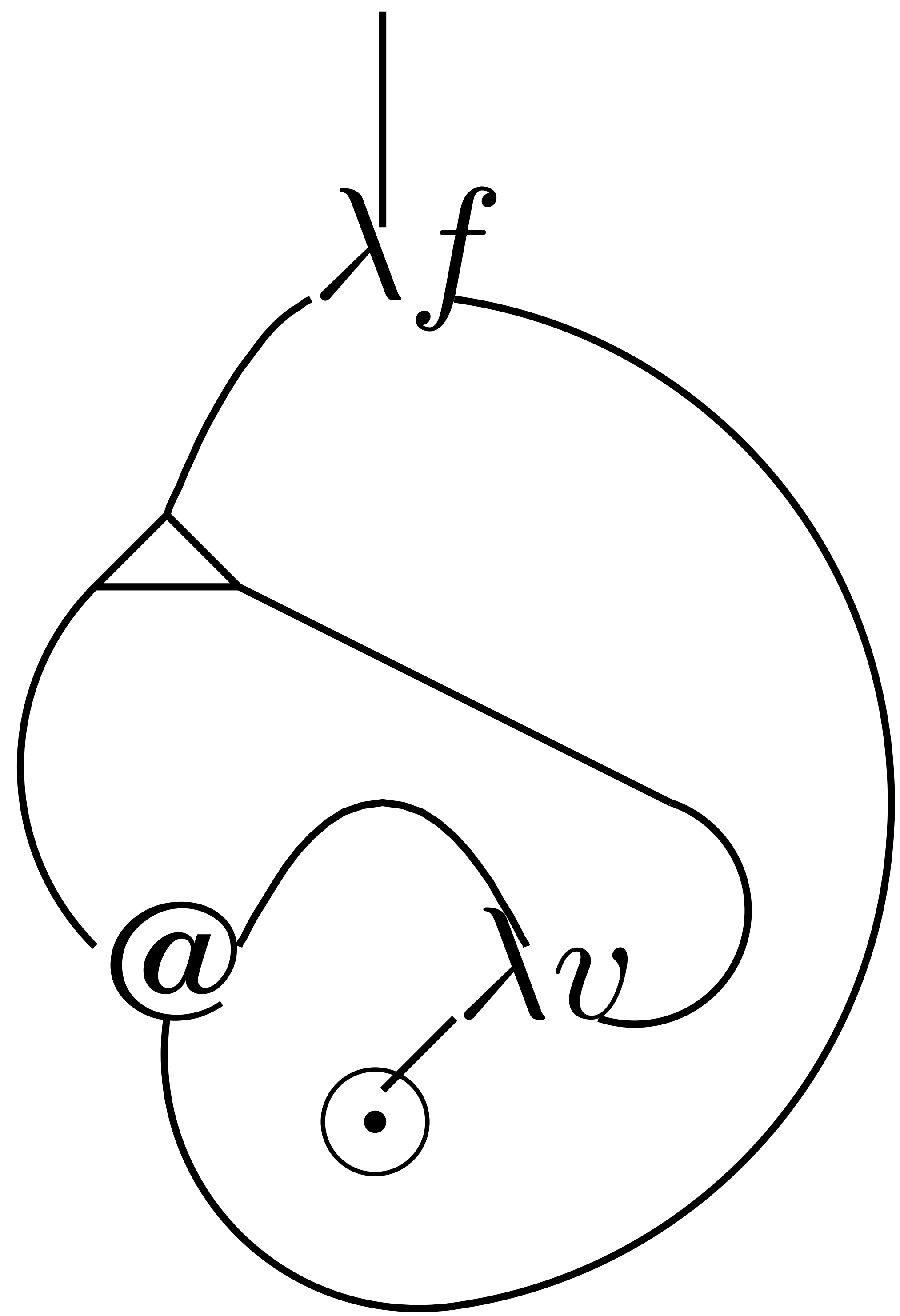
Every variable occurs once.



Every variable occurs once.

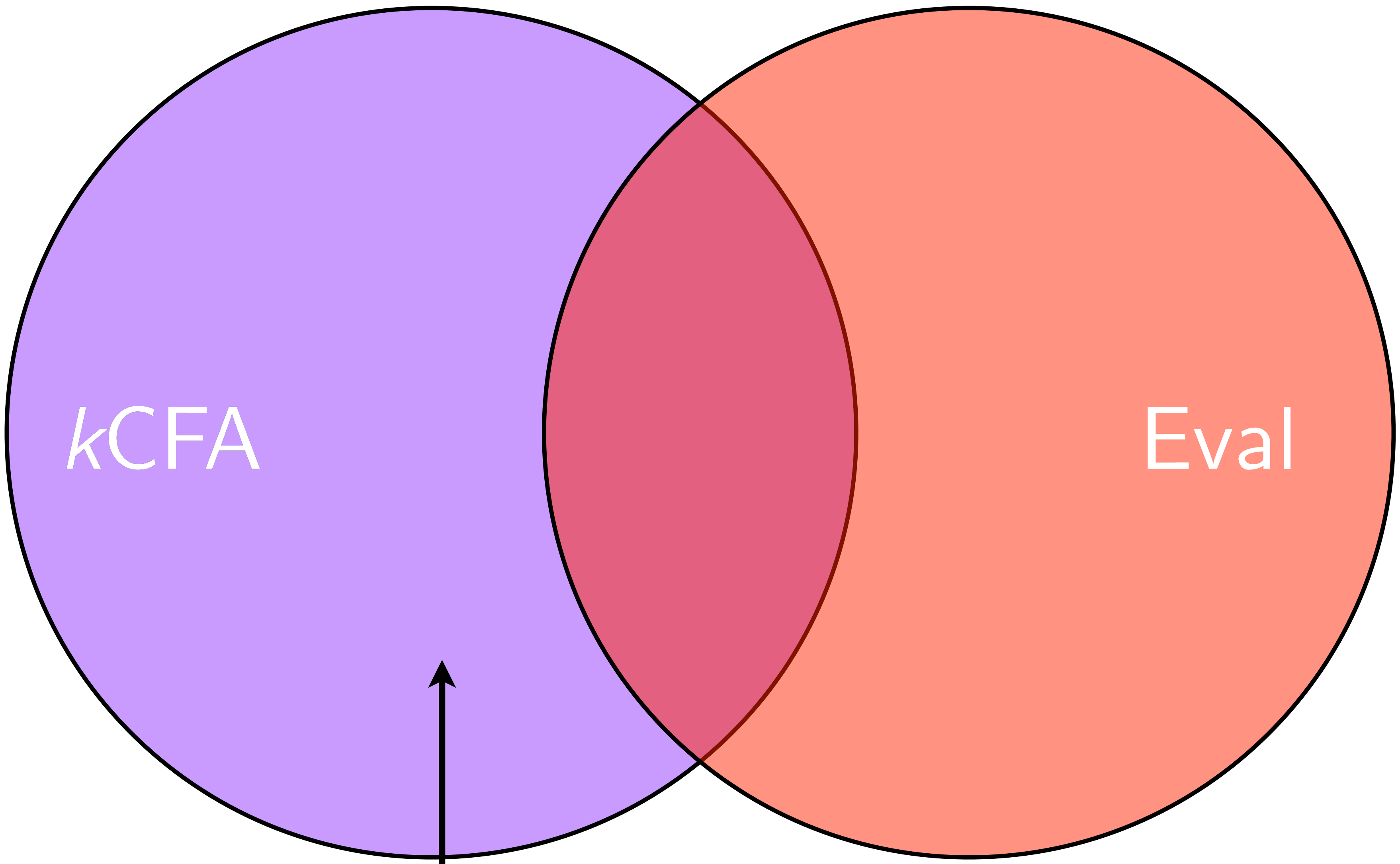

Sub0CFA
0CFA
Eval
Simple closure
Every variable occurs once.


Sub0CFA
0CFA
PTIME
Eval
Simple closure

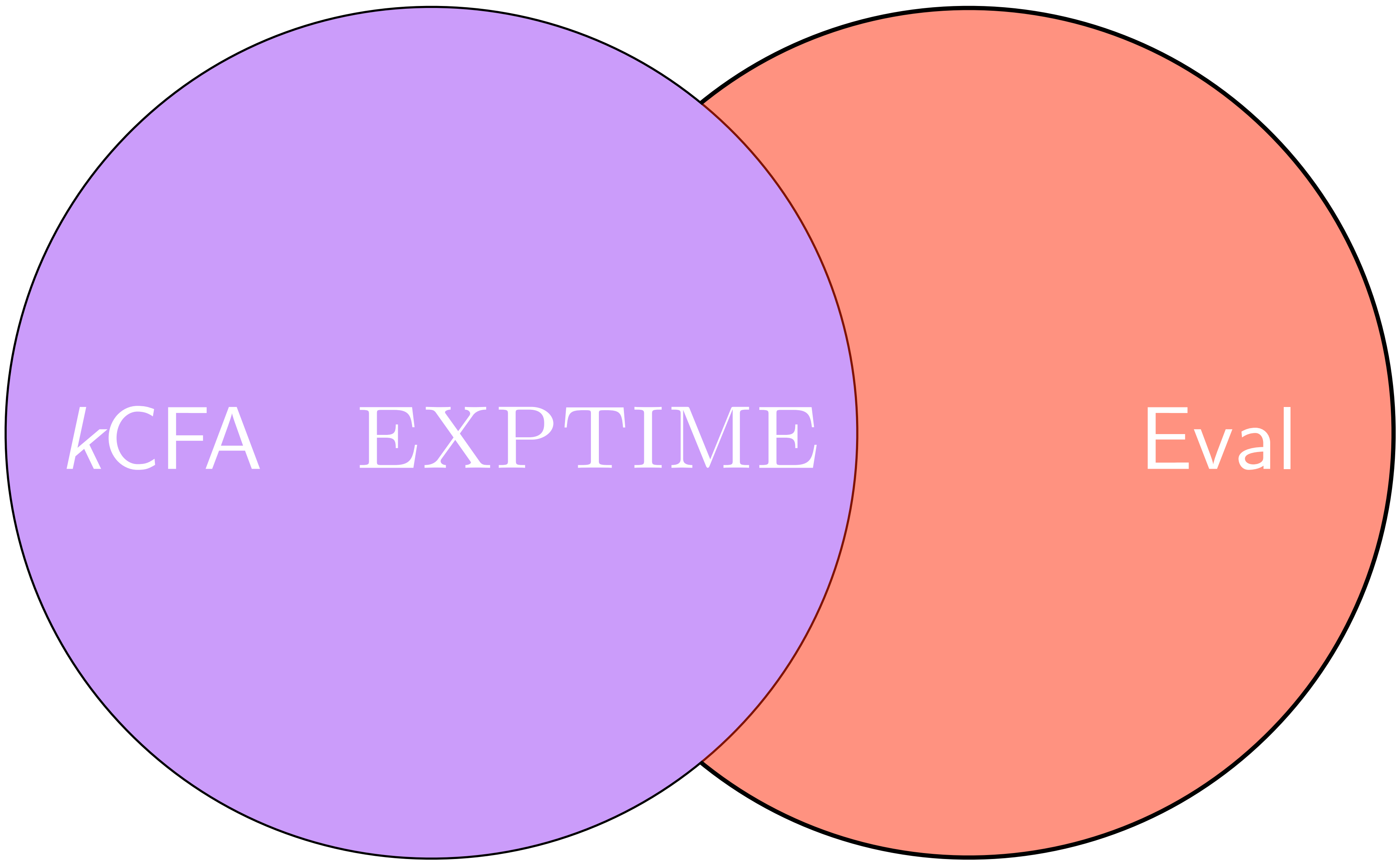


$[\![$`call/cc`$]\!]$

1CFA and EXPTIME



Datalog-style programming with analysis.



kCFA
EXPTIME
Eval

$$\widehat{\mathsf{C}} \in \widehat{\mathbf{Cache}} = \mathbf{Lab} \times \mathbf{Lab}^{\leq k} \to \mathcal{P}(\mathbf{Term} \times \mathbf{Env})$$

$$\begin{aligned}
\mathcal{A}[\![(t^{\ell_1} t^{\ell_2})^{\ell}]\!]^{ce}_{\delta} \quad = \quad & \mathcal{A}[\![t^{\ell_1}]\!]^{ce}_{\delta}; \mathcal{A}[\![t^{\ell_2}]\!]^{ce}_{\delta}; \\
& \textsf{foreach } \langle \lambda x.t^{\ell_0}, ce' \rangle \in \widehat{\mathsf{C}}(\ell_1, \delta) : \\
& \quad \hat{\mathsf{r}}(x, \lceil \delta\ell \rceil_k) \leftleftarrows \widehat{\mathsf{C}}(\ell_2, \delta); \\
& \quad \mathcal{A}[\![t^{\ell_0}]\!]^{ce'[x \mapsto \lceil \delta\ell \rceil_k]}_{\lceil \delta\ell \rceil_k}; \\
& \quad \widehat{\mathsf{C}}(\ell, \delta) \leftleftarrows \widehat{\mathsf{C}}(\ell_0, \lceil \delta\ell \rceil_k)
\end{aligned}$$

Hardness of kCFA relies on two insights:

1. Program points are approximated by an exponential number of closures.
2. *Inexactness* of analysis engenders *reevaluation* which provides *computational power*.

Many closures can flow to a single program point:

$$(\lambda w. w x_1 x_2 \ldots x_n)$$

- $\star$ n free variables
- $\star$ an *exponential* number of possible associated environments mapping these variables to program points (contours of length 1 in 1CFA).

Consider the following *non-linear* example

$$(\lambda f.(f\ \texttt{True})(f\ \texttt{False}))$$
$$(\lambda x.$$
$$(\lambda p.p(\lambda u.p(\lambda v.(\texttt{Implies}\ u\ v))))(\lambda w.wx))$$

Q: What does $\texttt{Implies}\ u\ v$ $\boxed{\text{evaluate to}}$?

A: $\texttt{True}$: it is equivalent to $\texttt{Implies}\ x\ x$, a tautology.

Q: What $\boxed{\text{flows out of}}$ $\texttt{Implies}\ u\ v$?

A: **both** $\texttt{True}$ and $\texttt{False}$: *Not true evaluation!*

$$(\lambda f_1.(f_1\ \mathtt{True})(f_1\ \mathtt{False}))$$
$$(\lambda x_1.$$
$$\quad(\lambda f_2.(f_2\ \mathtt{True})(f_2\ \mathtt{False}))$$
$$\quad(\lambda x_2.$$
$$\qquad(\lambda f_3.(f_3\ \mathtt{True})(f_3\ \mathtt{False}))$$
$$\qquad(\lambda x_3.$$
$$\qquad\qquad\ldots$$
$$\qquad\qquad(\lambda f_n.(f_n\ \mathtt{True})(f_n\ \mathtt{False}))$$
$$\qquad\qquad(\lambda x_n.$$
$$\qquad\qquad\quad E[(\lambda v.\phi\ v)(\lambda w.w x_1 x_2 \cdots x_n)])\cdots)))))$$

The idea:

- ⋆ Break machine ID into an exponential number of pieces
- ⋆ Do piecemeal transitions on pairs of puzzle pieces

$$\langle T, S, H, C, b \rangle$$

"At time T, machine is in state S, the head is at cell H, and cell C holds symbol b"

$\langle T, S, H, C, b \rangle$: *"At time T, machine is in state S, the head is at cell H, and cell C holds symbol b"*

1) Compute:

$$\delta\langle T, S, H, H, b\rangle\langle T, S', H', C', b'\rangle =$$
$$\langle T+1, \delta_Q(S,b), \delta_{LR}(S,H,b), H, \delta_\Sigma(S,b)\rangle$$

2) Communicate:

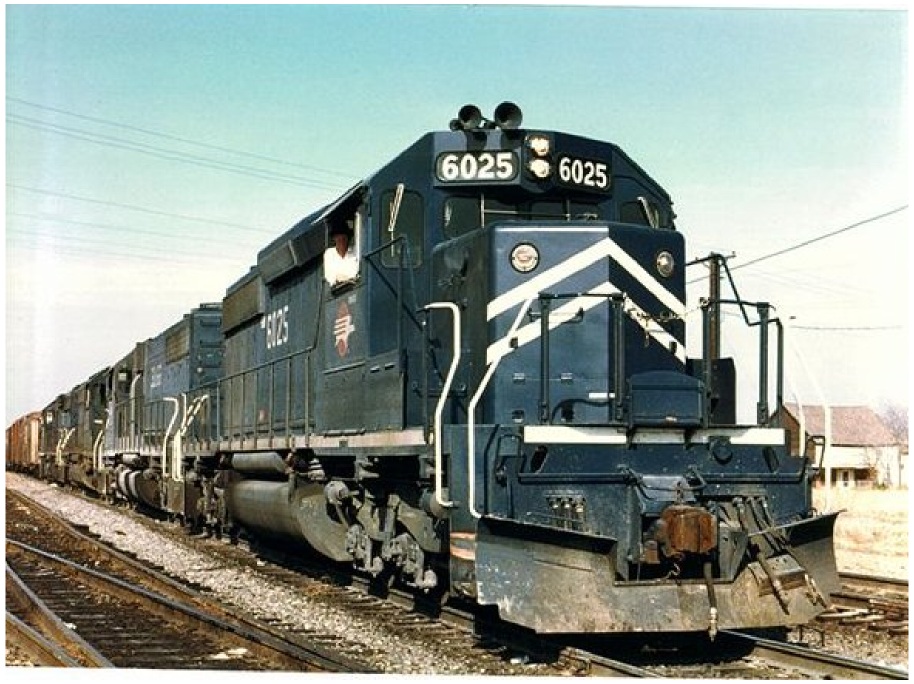
$$\delta\langle T+1, S, H, C, b\rangle\langle T, S', H', C', b'\rangle = \langle T+1, S, H, C', b'\rangle$$
$$(H' \neq C')$$

3) Otherwise:

$\delta\langle T, S, H, C, b\rangle\langle T', S', H', C', b'\rangle \; = \; \langle$some goofy  null value$\rangle$

$(T \neq T'$ and $T \neq T' + 1)$

Setting up initial ID, iterator, and test:

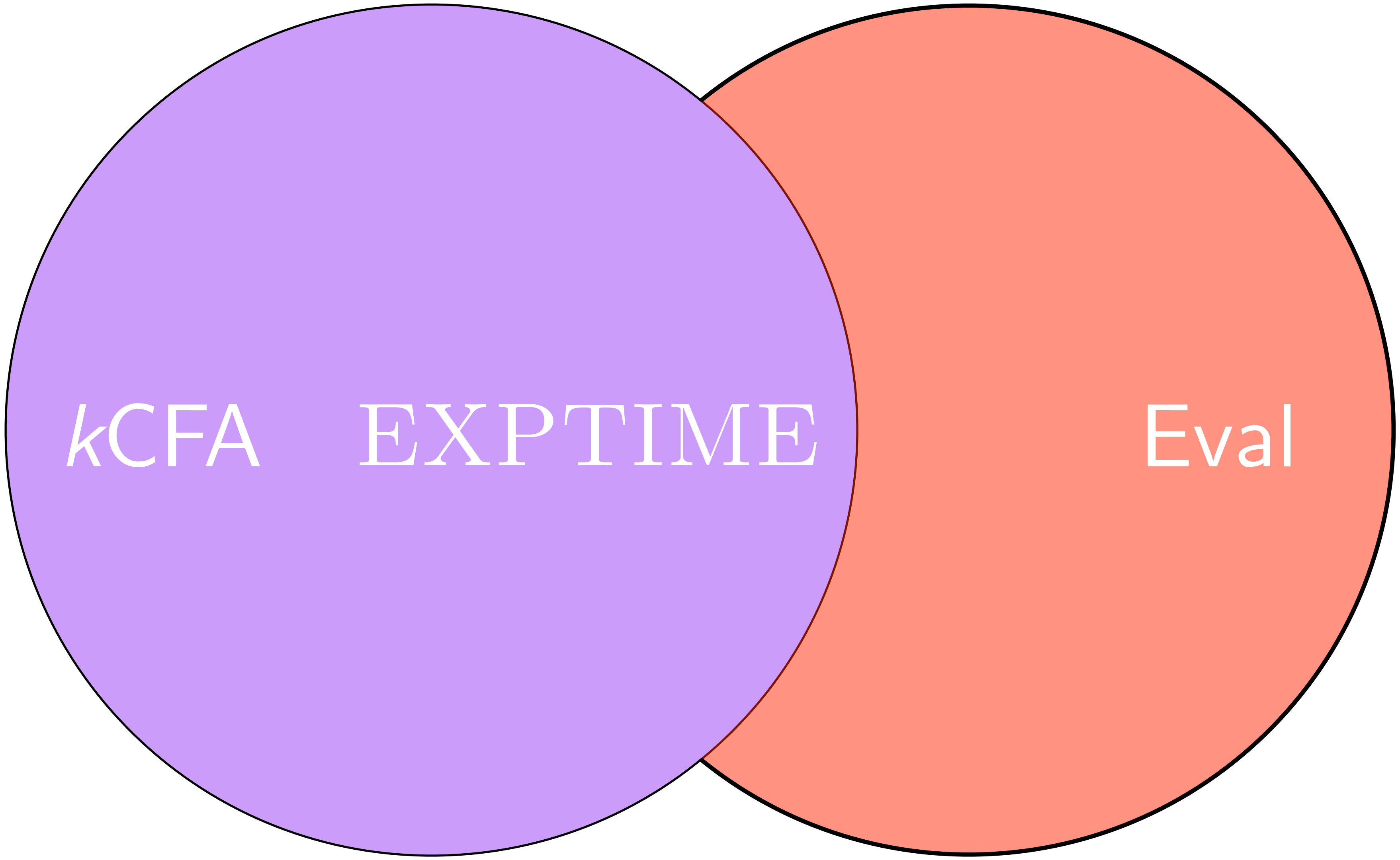


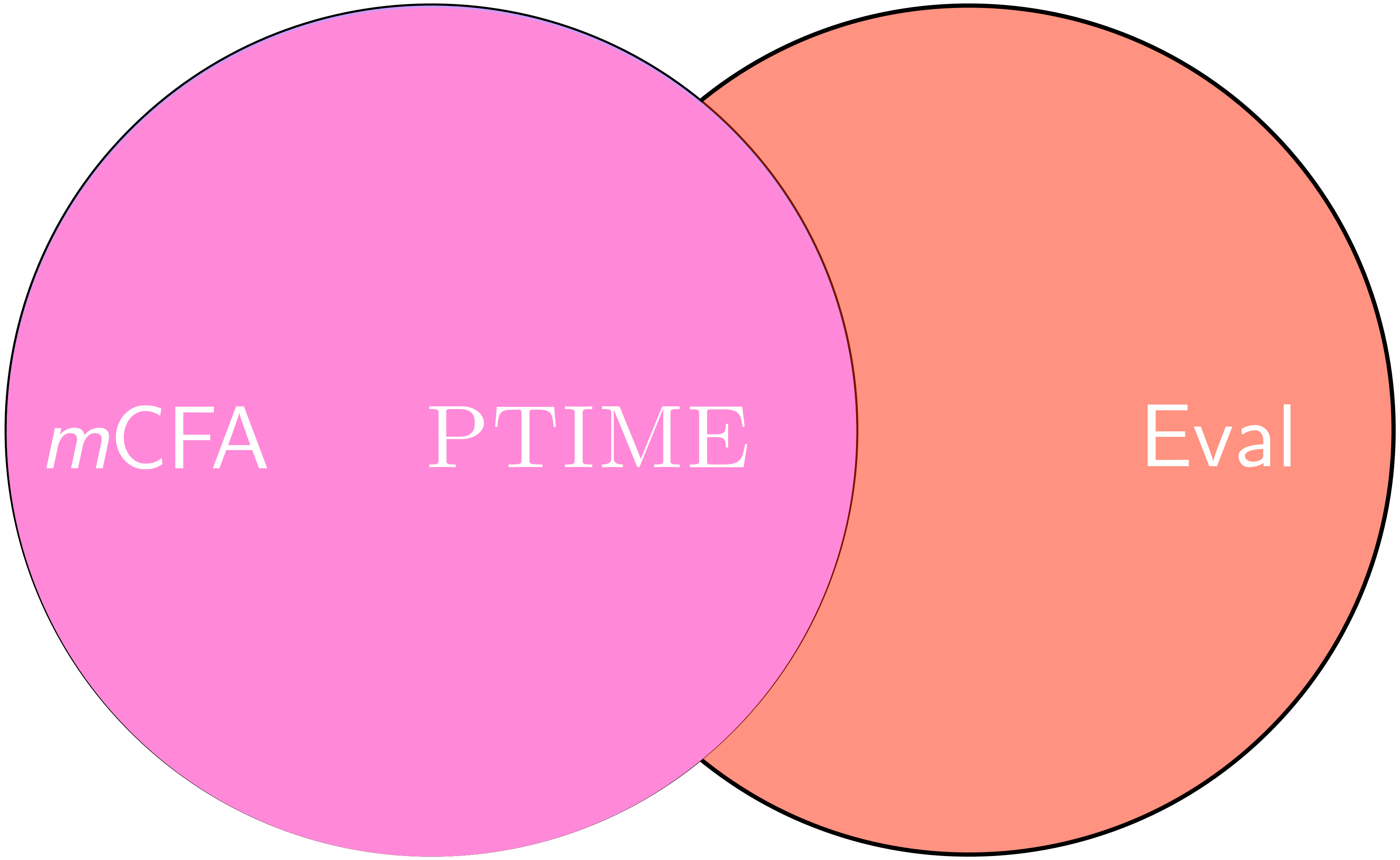
$$(\lambda f_1.(f_1\ \mathbf{0})(f_1\ \mathbf{1}))$$
$$(\lambda z_1.$$
$$\quad(\lambda f_2.(f_2\ \mathbf{0})(f_2\ \mathbf{1}))$$
$$\quad(\lambda z_2.$$
$$\qquad\ldots$$
$$\qquad\quad(\lambda f_N.(f_N\ \mathbf{0})(f_N\ \mathbf{1}))$$
$$\qquad\quad(\lambda z_N.$$

(let $\Phi =$ *coding of transition function of TM* in

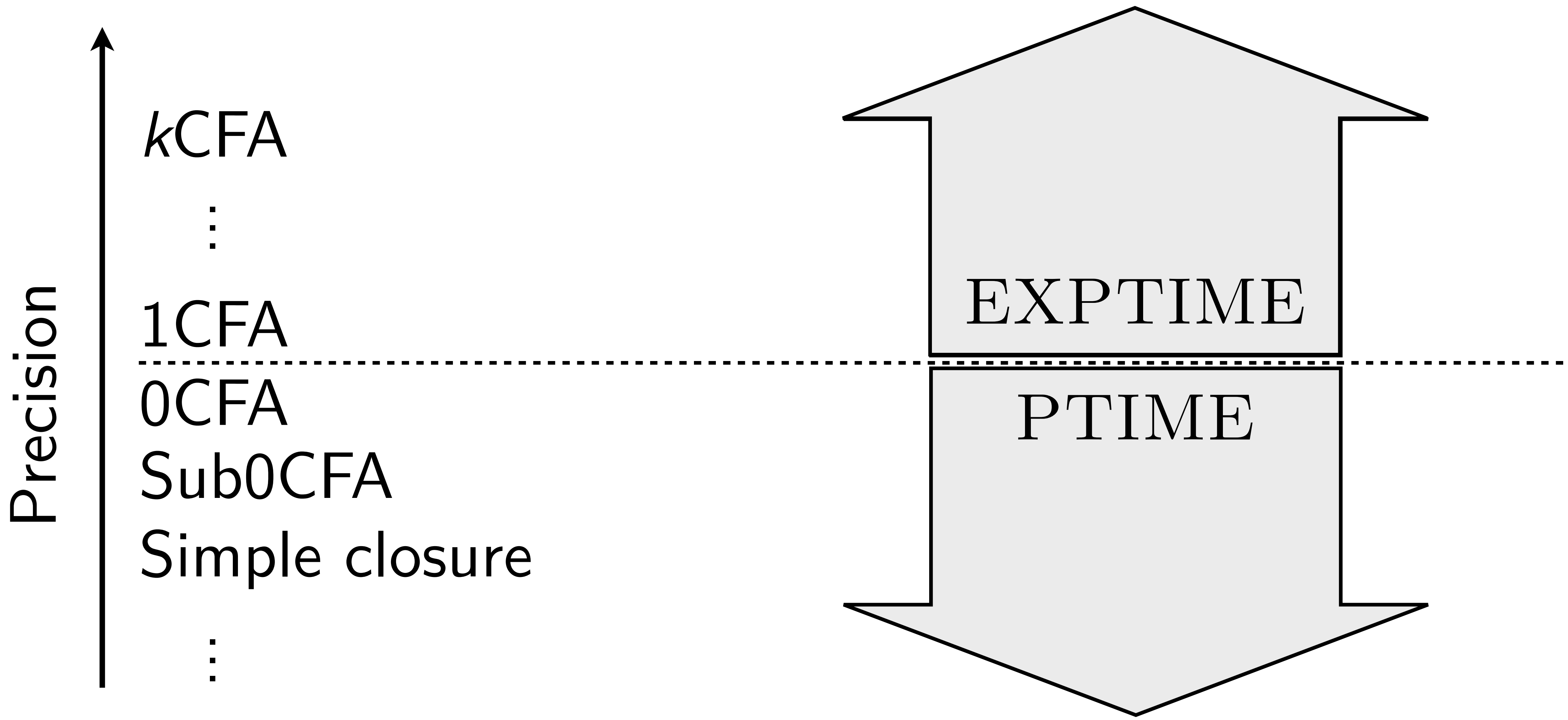
$\texttt{Widget}[\texttt{Extract}(Y\ \Phi\ (\lambda w.w\ \mathbf{0}\ldots\mathbf{0}\ Q_0\ H_0\ z_1 z_2\ldots z_N\ \mathbf{0}))])) \ldots))$

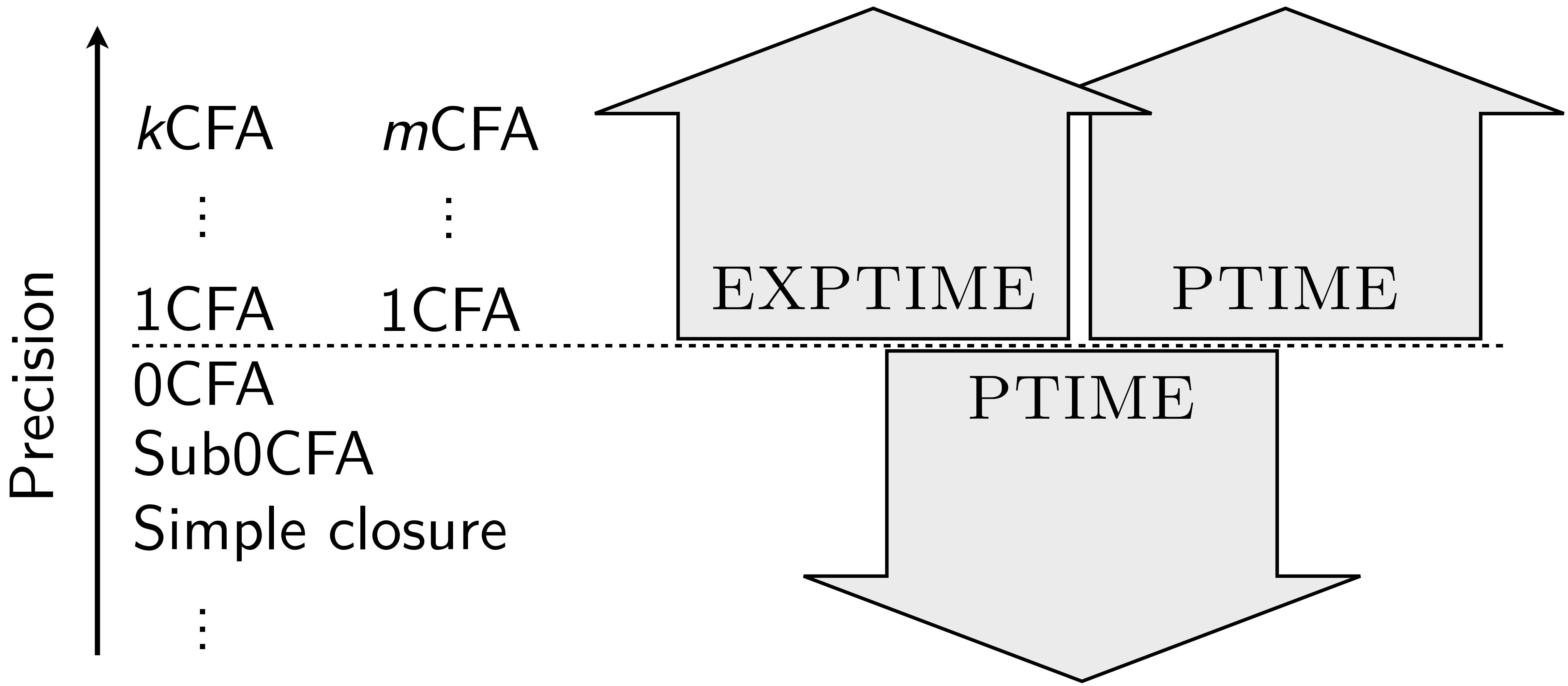
$\langle T, S, H, \qquad C, b\rangle$


*k*CFA
EXPTIME
Eval



Similar precision, better performance


Precision
kCFA
1CFA
0CFA
Sub0CFA
Simple closure
EXPTIME
PTIME


Precision
kCFA
mCFA
1CFA
1CFA
0CFA
Sub0CFA
Simple closure
EXPTIME
PTIME
PTIME