Complexity of Flow Analysis

David Van Horn and Harry Mairson

Overview

- $\star$ Monovariant flow analysis (0CFA) is complete for **PTIME**
  - — Analysis & evaluation are identical on linear programs
  - — 0CFA (and approximations) are inherently sequential
- $\star$ For any $k > 0$, kCFA is complete for time **EXPTIME**
  - — Exact characterization of complexity of kCFA hierarchy
  - — Validates observations that kCFA is intractable

Plan

- $\star$ Proving lower bounds — *programming with analysis*
  - — What is kCFA?
  - — Linearity and precision
  - — Non-linearity and an exponential iterator
- $\star$ Simulating exponential Turing machines with kCFA
- $\star$ Conclusions

Programming with analysis

A lower bound establishes the minimum computational requirements it takes to solve a class of problems.

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- ⋆ Subvert approximation: Analysis $\Rightarrow$ Evaluation
  - — For any abstract interpretation, there is a subset of the language on which abstract and concrete coincide.
- ⋆ Exploit approximation: Inexactness as combinatorial tool

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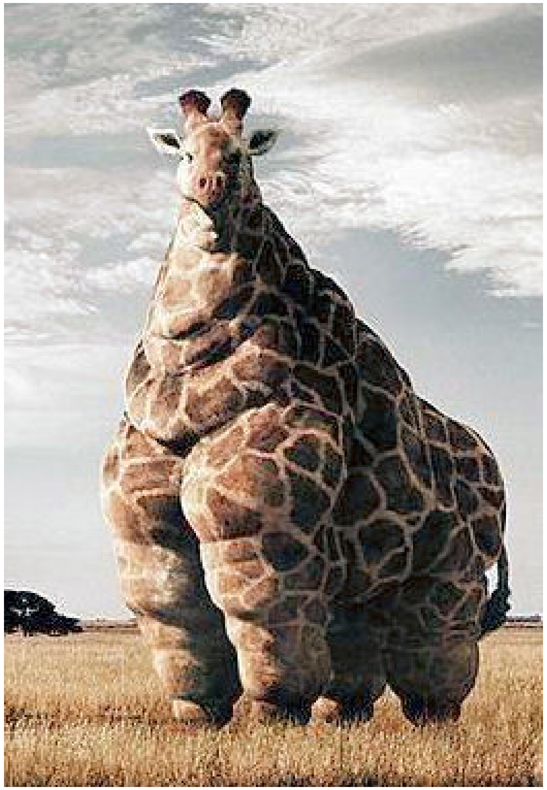
A (weird) compiler!

Strange animals

A compiler:

- ⋆ Source: exponential TMs with input
- ⋆ Target: the λ-calculus
- ⋆ Interpreter: kCFA (as TM simulator)

$\therefore$ kCFA is complete for **EXPTIME**.



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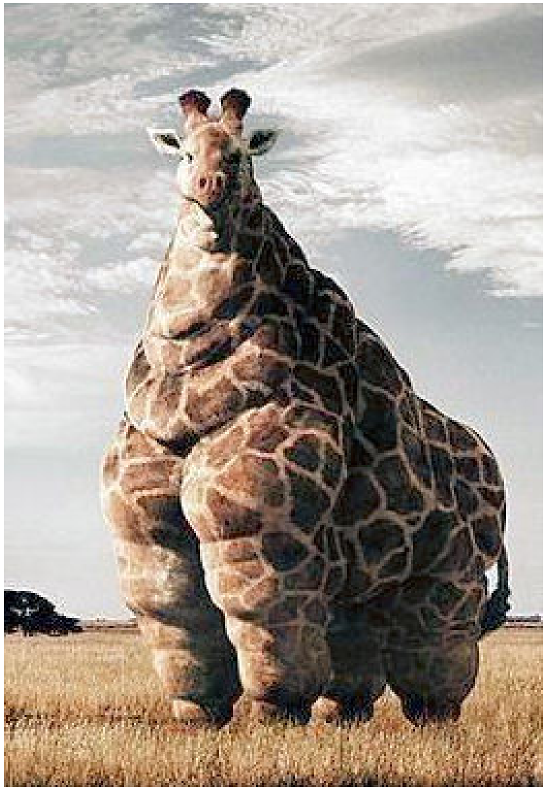

Another compiler:

- $\star$ Source: circuit with inputs
- $\star$ Target: the linear λ-calculus
- $\star$ Interpreter: 0CFA (as λ-evaluator)

$\therefore$ 0CFA is complete for **PTIME**.

More strange animals

Other compilers (ICFP'07):

- $\star$ Source: Boolean formulas
- $\star$ Target: the λ-calculus
- $\star$ Interpreter: kCFA (as SAT solver)

$\therefore$ kCFA is $\mathbf{NP}$-hard.

More strange animals

Other compilers (Mairson, POPL'90):

- $\star$ Source: exponential TMs with input
- $\star$ Target: ML
- $\star$ Interpreter: type inference (as ML evaluator)

$\therefore$ ML type inference is complete for **EXPTIME**.

More strange animals

Other compilers (Henglein and Mairson, POPL'91):

- $\star$ Source: elementary TMs with input
- $\star$ Target: System F_i
- $\star$ Interpreter: type inference (as System F_i evaluator)

$\therefore$ System F_i type inference is $\mathbf{DTIME}(\mathbf{K}(i,n))$-hard.

More strange animals

Other compilers (Mairson, JFP'04):

- $\star$ Source: circuit with inputs
- $\star$ Target: the linear λ-calculus
- $\star$ Interpreter: type inference (as λ-evaluator)

$\therefore$ Simple type inference is complete for **PTIME**.

More strange animals

Other examples (Neergaard and Mairson, ICFP'04):

- $\star$ Source: elementary TMs with input
- $\star$ Target: the λ-calculus
- $\star$ Interpreter: rank-k $\wedge$-type inference (as λ-evaluator)

$\therefore$ Rank-k $\wedge$-type inference is complete for $\mathbf{DTIME}(\mathbf{K}(k,n))$.

A complexity zoo of static analysis

0CFA $\equiv$ Simple closure analysis $\equiv$ Sub-0CFA $\equiv$ Simple type inference $\equiv$ Linear λ-calculus $\equiv$ MLL...

$\subset$

kCFA $\equiv$ ML type inference...

$\subset$

Rank-k intersection type inference...

$\subset$

Exact CFA $\equiv$ Simply typed λ-calculus...

$\subset$

∞CFA $\equiv$ The λ-calculus...

"Program analysis is still far from being able to precisely relate ingredients of different approaches to one another."

(Nielson et al. 1999)

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Flow analysis

Flow analysis is concerned with the sound approximation of run-time values at compile time.

Analysis answers decision problems such as:

does expression e possibly evaluate to value v?

- $\star$ The most approximate analysis always answers *yes*.
  — no resources to compute, but useless
- $\star$ The most precise analysis answers *yes* iff e evaluates to v.
  — useful, but unbounded resources to compute

For tractability, there is a necessary sacrifice of information in static analysis. (A blessing and a curse.)

kCFA

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It did not take long to discover that the basic analysis, for any $k > 0$, was intractably slow for large programs.

Shivers, Higher-order control-flow analysis in retrospect: Lessons learned, lessons abandoned (2004)

Polyvariance

During reduction, a function may copy its argument:

$$((\lambda f. \cdots (f e_1)^{\ell_1} \cdots (f e_2)^{\ell_2} \cdots)(\lambda x.e))$$

Contours (strings of application labels) let us talk about e in each of the distinct calling contexts.

Cache-based evaluator

Exp	$e ::= t^{\ell}$	expressions (or labeled terms)
Term	$t ::= x \mid e\, e \mid \lambda x.e$	terms (or unlabeled expressions)

Evaluate the term t, which is closed under environment ce.

$$\mathcal{E}[\![t^{\ell}]\!]^{ce}_{\delta}$$

Write the result into location (ℓ, δ) of the cache C.

$\mathsf{C}(\ell, \delta) = v$ means t^{ℓ} evaluates to v in context δ.

∞CFA

A cache-based evaluator:

$$\mathsf{C} \in \mathbf{Cache} = (\mathbf{Lab} + \mathbf{Var}) \times \mathbf{Lab}^\star \to (\mathbf{Term} \times \mathbf{Env})$$

$$\begin{array}{rcl}
\mathcal{E}[\![(t^{\ell_1} t^{\ell_2})^{\ell}]\!]^{ce}_{\delta} & = & \mathcal{E}[\![t^{\ell_1}]\!]^{ce}_{\delta}; \mathcal{E}[\![t^{\ell_2}]\!]^{ce}_{\delta}; \\
 & & \text{let } \langle \lambda x.t^{\ell_0}, ce' \rangle = \mathsf{C}(\ell_1, \delta) \text{ in} \\
 & & \quad \mathsf{C}(x, \delta\ell) \leftarrow \mathsf{C}(\ell_2, \delta); \\
 & & \quad \mathcal{E}[\![t^{\ell_0}]\!]^{ce'[x \mapsto \delta\ell]}_{\delta\ell}; \\
 & & \quad \mathsf{C}(\ell, \delta) \leftarrow \mathsf{C}(\ell_0, \delta\ell)
\end{array}$$

kCFA

An *abstraction* of the cache-based evaluator:

$$\widehat{\mathsf{C}} \in \widehat{\mathbf{Cache}} = (\mathbf{Lab} + \mathbf{Var}) \times \mathbf{Lab}^{\leq k} \to \mathcal{P}(\mathbf{Term} \times \mathbf{Env})$$

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& & \quad \widehat{\mathsf{C}}(x) \leftarrow \widehat{\mathsf{C}}(\ell_2); \\
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(An evaluator for linear λ-calculus).

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Linearity and evaluation

Since in a *linear* λ-term,

- $\star$ each abstraction can be applied to at most one argument
- $\star$ each variable can be bound to at most one value

Analysis of a linear term coincides exactly with its evaluation.

$$((\lambda f. \cdots (f e_1) \cdots (f e_2) \cdots)(\lambda x.e))$$

Boolean logic

Coding Boolean logic in linear λ-calculus:

$$\begin{aligned}\texttt{TT} &\equiv \lambda p.\text{let } \langle x, y\rangle = p \text{ in } \langle x, y\rangle & \texttt{True} &\equiv \langle \texttt{TT}, \texttt{FF}\rangle \\ \texttt{FF} &\equiv \lambda p.\text{let } \langle x, y\rangle = p \text{ in } \langle y, x\rangle & \texttt{False} &\equiv \langle \texttt{FF}, \texttt{TT}\rangle\end{aligned}$$

$$\texttt{Copy} \equiv \lambda b.\text{let } \langle u, v\rangle = b \text{ in } \langle u\langle \texttt{TT}, \texttt{FF}\rangle, v\langle \texttt{FF}, \texttt{TT}\rangle\rangle$$

$$\begin{aligned}\texttt{Implies} \equiv\ & \lambda b_1.\lambda b_2. \\ & \text{let } \langle u_1, v_1\rangle = b_1 \text{ in} \\ & \text{let } \langle u_2, v_2\rangle = b_2 \text{ in} \\ & \text{let } \langle p_1, p_2\rangle = u_1\langle u_2, \texttt{TT}\rangle \text{ in} \\ & \text{let } \langle q_1, q_2\rangle = v_1\langle \texttt{FF}, v_2\rangle \text{ in} \\ & \quad \langle p_1, q_1 \circ p_2 \circ q_2 \circ \texttt{FF}\rangle\end{aligned}$$

0CFA and PTIME

$\lambda e_1.\lambda e_2.\lambda e_3\lambda e_4.\lambda e_5.\lambda e_6.$
  `Andgate` $e_2\ e_3\ (\lambda e_7.$
  `Andgate` $e_4\ e_5\ (\lambda e_8.$
  `Copygate` $f\ (\lambda e_9.\lambda e_{10}.$
  `Orgate` $e_1\ e_9\ (\lambda e_{11}.$
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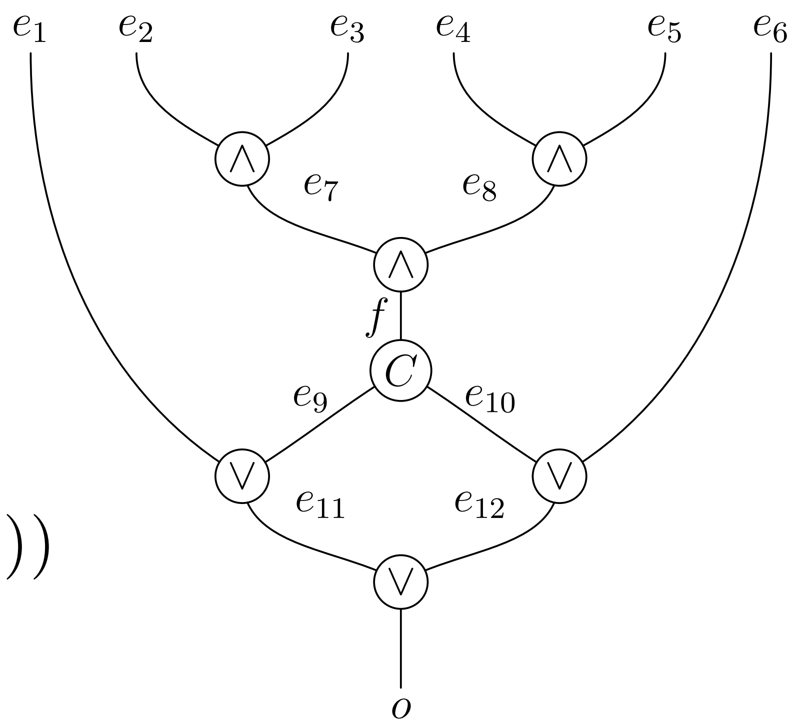

Theorem 0CFA decision problem is complete for PTIME.

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Approximation as power tool

Hardness of kCFA relies on two insights:

1. Program points are approximated by an exponential number of closures.
2. *Inexactness* of analysis engenders *reevaluation* which provides *computational power*.

Abstract closures

Many closures can flow to a single program point:

$$(\lambda w.w x_1 x_2 \ldots x_n)$$

- $\star$ n free variables
- $\star$ an *exponential* number of possible associated environments mapping these variables to program points (contours of length 1 in 1CFA).

Toy calculation, with insights

Consider the following *non-linear* example

$$(\lambda f.(f\ \mathtt{True})(f\ \mathtt{False}))$$
$$(\lambda x.$$
$$(\lambda p.p(\lambda u.p(\lambda v.(\mathtt{Implies}\ u\ v))))(\lambda w.wx))$$

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We are *computing with the approximation* (spurious flows).

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Jigsaw puzzles, Machines

The idea:

- ⋆ Break machine ID into an exponential number of pieces
- ⋆ Do piecemeal transitions on pairs of puzzle pieces

$$\langle T, S, H, C, b \rangle$$

"At time T, machine is in state S, the head is at cell H, and cell C holds symbol b"

Jigsaw puzzles, Machines

$\langle T, S, H, C, b\rangle$: *"At time T, machine is in state S, the head is at cell H, and cell C holds symbol b"*

1) Compute:

$$\delta\langle T, S, H, H, b\rangle\langle T, S', H', C', b'\rangle =$$
$$\langle T+1, \delta_Q(S, b), \delta_{LR}(S, H, b), H, \delta_\Sigma(S, b)\rangle$$

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2) Communicate:

$$\delta\langle T+1, S, H, C, b\rangle\langle T, S', H', C', b'\rangle = \langle T+1, S, H, C', b'\rangle$$
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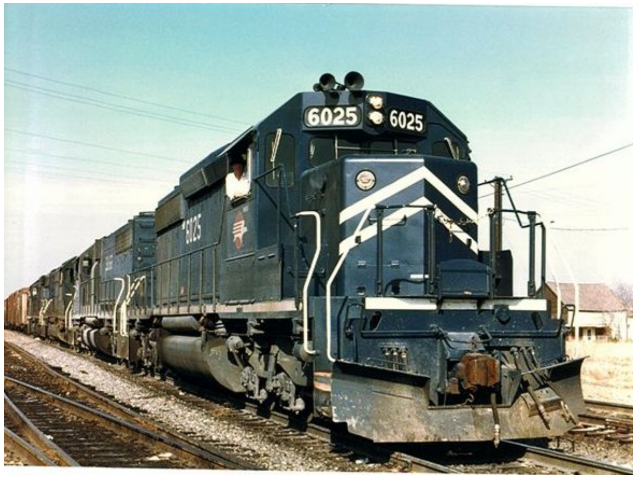
3) Otherwise:

$\delta\langle T,S,H,C,b\rangle\langle T',S',H',C',b'\rangle = \langle$some goofy  null value$\rangle$

$$(T \neq T' \text{ and } T \neq T'+1)$$

The real deal



Setting up initial ID, iterator, and test:

$$(\lambda f_1.(f_1\ \mathbf{0})(f_1\ \mathbf{1}))$$
$$(\lambda z_1.$$
$$(\lambda f_2.(f_2\ \mathbf{0})(f_2\ \mathbf{1}))$$
$$(\lambda z_2.$$
$$\ldots$$
$$(\lambda f_N.(f_N\ \mathbf{0})(f_N\ \mathbf{1}))$$
$$(\lambda z_N.$$

(let $\Phi =$ *coding of transition function of TM* in

$$\texttt{Widget}[\texttt{Extract}(Y\ \Phi\ (\lambda w.w\ \underbrace{\mathbf{0}\ldots\mathbf{0}\ Q_0\ H_0\ z_1 z_2 \ldots z_N\ \mathbf{0}}_{\langle T, S, H, \quad C, b\rangle}))])) \ldots))$$

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$$\langle T, S, H, \qquad C, b\rangle$$

$$\Phi \equiv (\lambda p.p(\lambda x_1.\lambda x_2.\ldots.\lambda x_m.p(\lambda y_1.\lambda y_2 \ldots \lambda y_m.$$
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Theorem kCFA decision problem is complete for **EXPTIME**.

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Work harder to learn less.

Doggie bag

- ⋆ Linearity is key in understanding static analysis
- ⋆ 0CFA is inherently sequential
- ⋆ There is no tractable algorithm for kCFA
- ⋆ The approximation of kCFA is what makes it hard

The End

Thank you.