Ordering Multiple Continuations on the Stack

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Note This technical report accompanies our publication in PEPM 2011 with the same name. In this report, we give a more general definition of Restricted CPS that does not require the image of call/cc for first-class control. We also provide a proof of the theorem for the stackability of continuation arguments.

Abstract

Passing multiple continuation arguments to a function in CPS form allows one to encode a wide variety of direct-style control constructs, such as conditionals, exceptions, and multi-return function calls. We show that, with a simple syntactic restriction on the CPS language, one can prove that these multi-continuation arguments can be compiled into stack frames in the traditional manner. The restriction comes with no loss in expressive power, since we can still encode the same control mechanisms.

In addition, we show that tail calls can be generalized efficiently for many continuations because the run-time check to determine which continuation to pop to can be avoided with a simple static analysis. A prototype implementation in Scheme48 shows that our analysis is very precise.

1. Introduction

Continuation-passing style (CPS) has a long history as a compiler intermediate representation [1, 5, 7, 11], going back to Steele’s Rabbit compiler [14]. More recently, Kennedy studied the engineering benefits afforded by CPS-based intermediate representations [8]. When used in compilers, CPS is usually extended in two ways from the simple form we see in more foundational developments [8].

First, every element of a CPS term (lambdas, variable references, and calls) is statically marked as either a “user” or a “continuation” term. There is a similar user/continuation partition among dynamic values, which respects the static partition: continuation values are produced only from “continuation” λ terms, bound only to “continuation” variables, and invoked only at “continuation” call sites; likewise for “user” values. This partition enables the compiler to produce code that uses a stack to manage procedure calls. Continuations are simply procedures whose environment record is a stack frame.

Second, CPS-based compilers often pass many continuations across function calls:

- The SML/NJ compiler implements exceptions by passing two continuations to each function: one for the normal return point and one for the current exception handler.
- ORBIT [8] encodes conditionals as primitives that take two continuations. Instead of having special syntax for if/then/else, ORBIT employs a primitive procedure, \( \lambda x \rightarrow \text{if } x \), with three arguments, a boolean and two continuations:
  \[
  (\lambda x \rightarrow \text{bool cont}_{\text{then}} \ cont_{\text{else}})
  \]
  Control branches to \( \text{cont}_{\text{then}} \) if the boolean argument is true, and to \( \text{cont}_{\text{else}} \) if it is false. By representing control operators as functions, the compiler can wring even more utility out of its general capabilities for reasoning about functions. This technique has been explored in detail in the literature [12].
- The “multi-return λ calculus” (λMR) [12] can be considered a generalization of the aforementioned Orbit technique. Where Orbit uses this technique internally, in λMR the mechanism is exported to the language level: the direct-style term language is designed to provide the power of passing multiple return points to user procedures, yet ensure that these return points can be stack allocated. The multi-return mechanism was specifically designed to fit naturally with an IR that uses multiple continuations to represent the multiple return points.
- Finally, multiple-continuation function calls can be used to implement “stream processing” computations, such as DSP pipelines [13].

These two extensions are a standard part of the lore of engineering compilers using CPS. However, they raise issues that have yet to be addressed. First, compiler writers work on the assumption that statically partitioned CPS allows continuation closures to be treated as stack frames. The question arises: why is this a safe assumption?

If we only have single-continuation calls, then it is fairly simple to show that the continuation environment records can be managed with a LIFO policy. But what happens when, as is often the case in practice, we pass multiple continuations across procedure calls? The compiler assumes that the continuations being passed all lie on the same stack, and so can all be passed as pointers into that stack. Is this in fact always true? Does it remain true when one lifts the idea of multiple return points from a limited, compiler-internal technique to a general user mechanism, as in λMR?

Further, when a function call is made, tail recursion requires that the compiler clear the stack back to the caller’s continuation. Even if it is safe to suppose that all continuation arguments point into the same stack, the compiler must now pop the stack back to the most recent of these continuations. At a fixed call site, the continuation that is the “high-water” one can vary dynamically from call to call; in these cases, the compiler must emit code to compare the continuations at run time, in order to correctly adjust the stack.

This paper addresses the issues raised by the demands of stack-managing procedure calls in a CPS setting that permits multiple continuations to be passed across function calls.

- First, we describe a reasonable static restriction on CPS that ensures that multiple continuations can be safely passed across function calls as pointers into a single stack (sections 2-5).
• Second, we describe a higher-order flow analysis that resolves
the order of various continuations on a common stack, permit-
ting a program to avoid computing the “high water” continuation
at run time. This helps to make complex multi-return-point
program structure a more pay-as-you-go feature (section 5).
• Third, we develop λ_MR as a motivating example: it can be
naturally converted into our Restricted CPS form using multiple
continuations; these continuations can be safely represented
as stack frames; and λ_MR programs that call procedures with
many return points can be analyzed by our flow analysis.
It’s worth noting that, while Fisher and Shivers developed λ_MR
with an eye towards representing programs written in it with
multi-continuation CPS, they did not exhibit a CPS conversion
algorithm for their language. The conversion we show is inter-
esting in that it handles the issue of “control polymorphism”
that arises by means of a simple type system; the CPS conver-
sion is thus type-directed (section 7).
• Fourth, we report on experimental results from a prototype
implementation of our analysis in Scheme48. Our findings show
that the analysis can find the youngest continuation in most
cases, and it requires little increase in compilation time and
implementation effort over k-CFA (section 8).

These results are obtained in a setting that permits continuations
to be captured by operators such as call/cc, which can force the
run-time stack to be copied to, and restored from, the heap. The
net result is to put CPS intermediate representations as they are
employed in practice onto a more solid footing, and to make multi-
continuation function calls more efficient, in a general setting.

2. Restricted CPS

We propose Restricted CPS as a variant of Partitioned CPS [7].
Partitioned CPS splits the variables, lambdas and calls of a CPS
program into disjoint sets, the “user” and the “continuation” set,
so that it is easy to distinguish the two syntactically. Elements of
the direct-style source program end up in the “user” set in CPS.
Continuations and calls added by the CPS transform end up in the
“continuation” set.

We begin with a brief description of Partitioned CPS (Fig. 1).
The partitioning between the user and the continuation world hap-
pens by assigning labels to CPS terms from two disjoint sets: user
elements get labels from ULab and continuation elements get la-
bel from CLab. Hence, UVar, UCall and CLam refer to user
variables, lambdas and calls respectively. Similarly, CVar, CLam
and CCall refer to continuation variables, lambdas and calls.

We assume that all variables in a program have distinct names
and all labels are unique. In such a program, the function \( VL(v) \)
returns the label of the lambda that binds the variable \( v \) and \( LV(l) \)
returns the list of continuation parameters of the \( ulam \) labeled \( l \).
For a lambda term \( lam \), \( FP(lam) \) is its set of formal parameters.
The function \( FV(h) \) returns the set of free variables of the term \( h \).
We write \( u\_ulam_\gamma(h) \) for the innermost user lambda that contains
the term \( h \) as a subterm. Concrete syntax enclosed in \[ \] denotes
an item of abstract syntax.

We use two notations for tuples, \( (e_1, \ldots, e_n) \) and \( \langle e_1, \ldots, e_n \rangle \),
to avoid confusion when tuples are deeply nested. We use the latter
for lists as well; ambiguities are resolved by the context. Lists are
also described by a head-tail notation, e.g., \( 3 :: (1, 3, -47) \).

User functions take any number of user arguments and one
or more continuation arguments. Continuation functions take only
user arguments. In CPS, “returning” happens by calling a contin-
uation. Hence, only \( ulams \) can be returned, not \( clams \). Thus, a con-
tinuation can only escape when it is bound to a \( cvar \) that occurs
in a \( ulam \).

\[
\begin{align*}
pr \in PR & \quad ::= \quad \{(\lambda(halt) callD)\} \\
v \in Var & \quad = \quad UVar + CVar \\
u, u\_var \in UVar & \quad = \quad a \ set \ of \ identifiers \\
k, c\_var \in CVar & \quad = \quad a \ set \ of \ identifiers \\
lam \in Lam & \quad = \quad Ulam + CLam \\
ulam \in ULab & \quad ::= \quad \{(\lambda_1(u^k) callD)\} \\
clam \in CLam & \quad ::= \quad \{(\lambda_1(u^k) callD)\} \\
ccall & \quad ::= \quad UCall + CCall \\
ucall & \quad ::= \quad (f e q^+) \\
ccall & \quad ::= \quad (f e^+ q^+) \\
h \in Exp & \quad = \quad UExp + CExp \\
f, e \in UExp & \quad = \quad UVar + ULMam \\
q \in CExp & \quad = \quad CVar + CLam \\
\psi \in Lab & \quad = \quad ULab + CLab \\
l \in ULab & \quad = \quad a \ set \ of \ labels \\
\gamma, \zeta \in CLab & \quad = \quad a \ set \ of \ labels
\end{align*}
\]

Figure 1. Partitioned CPS

\[
\begin{align*}
\text{(define (square n cc h)} \\
\text{(number? n (lambda (test)} \\
\text{(if test} \\
\text{(lambda (a2 (* n n cc))} \\
\text{(lambda (a3 (h "not a number")))))}
\end{align*}
\]

Figure 2. Non-local exit

Many applications of multiple continuations use them in a
“downward” fashion: after its creation, a continuation is passed
as an argument to a number of \( ulams \) and then called — it is never
captured in a user closure. Thus, a simple syntactic constraint can
forbid first-class control: a \( ulam \) may only refer to continuations
from its list of formals, it cannot have free \( cvars \).

We want to allow first-class control, but in a way that per-
mits effective reasoning about the stack behavior of continuations.
Therefore, we propose another syntactically-restricted variant of
CPS, which we call “Restricted Continuation-Passing Style” (ab-
brev. RCS).

Definition 1 (Restricted CPS). A program is in Restricted CPS iff a
continuation variable can appear free in a user lambda in operator
position only.

With this definition, continuations escape in a well-behaved way:
a continuation can only be called after its escape, it cannot be passed
as an argument again. The CPS translation of call/cc, which is
\( (\lambda f f c (\lambda (x k) (cc x)) cc) \), is valid in RCS. In section 5
we show that even in the presence of call/cc the continuation
arguments of a \( ulam \) are still on the stack.

The simple function in Fig. 2 takes two continuations. It com-
putes the square of its argument and passes it to the current con-
tinuation, or it calls the handler continuation if it is passed a non-
number. The program is in RCS since the user functions only refer
to continuations that are passed to them1.

3. Stack management in RCPS

Might and Shivers [2] generalized ORBIT’s stack policy to handle
multiple continuations. Here, we give an outline of this stack policy.

At run time, continuations are closures whose environments live
on the stack. A continuation is represented as a pair \( (c, s) \) where
\( c \) is a pointer to its code and \( s \) a pointer into the stack. Continuations

\[\text{Note that although we use the } \lambda\text{-calculus to develop our theory, we add}\]
\[\text{constants and primitives in the examples to keep them short and clear.} \]
access their free variables from a pointer into the stack, never from the heap. To ensure this in the presence of first-class continuations, we have to copy the stack when a continuation escapes and restore it later when it is called.

Before a call to a user function \([\{ f \ e_1 \ldots e_n \ q_1 \ldots q_m \}]\), we want to retain the frames needed for \(q_1 \ldots q_m\) and remove any redundant frames. There are two possibilities:

- In a *tail call*, all \(q_j\)s are variables, so they are bound to closures already born. The frames pushed after the birth of the youngest closure are not needed. We pop these frames to restore the stack to the environment of the youngest closure. This way, all continuations are retained when we enter \(f\).

- In a *non tail call*, some \(q_j\)s are lambdas. These are newly born continuation closures, closed over the current stack pointer. Thus, all frames are needed and we leave the stack intact.

After this adjustment, the environment of the youngest continuation is at the top of the stack. We push a frame for \(f\)’s arguments and jump to \(f\). Generally, this policy maintains the following invariant: when a \(\lambda ulam\) is executing, the second frame is the youngest live continuation.

In the same spirit, before calling a continuation \([\langle q \ e^* \rangle]\), its environment must be on the top of the stack, so we reset the stack to the stack of its birth. We then push a frame for its arguments and jump to \(q\). The invariant maintained here is that during \(q\)’s execution the second frame points to its environment.

Returning to our example, if we run \((\text{square} 5 \text{ halt err})\) the actions on the stack are \(\langle\text{square}\rangle \langle\text{number?}\rangle \langle\text{number?}\rangle \langle 1 \rangle \langle 21 \ 2 \ 1 \rangle \langle\text{square}\rangle \langle* \*\rangle \langle\text{halt}\rangle\). The notation \(\langle\psi\rangle\) means pushing a frame for \(\lambda \psi\), and \(\langle\psi\rangle\) pops it. Initially we push frames for \(\text{square}\) and \(\text{number?}\). When we evaluate \(\lambda \psi\) we pop a frame to restore the stack of its birth and then push a frame for its argument. The execution continues along these lines. The only thing to note is the evaluation of \(\langle* \ n \ n \ cc\rangle\); \(cc\) is bound to \(\text{halt}\), so to maintain the stack invariant we have to pop to the stack at the time of \(\text{halt}\)’s birth. Thus, we pop three frames before pushing \(\langle*\rangle\).

4. Frame strings

In order to formally express stack properties and prove them, we must have a way to describe actions on the stack. In languages without tail calls, these push and pop actions correspond to sequences of calls and returns that nest properly. The call-string mechanism \([10]\) can be used to describe these sequences. However, in properly tail-recursive languages calls and returns no longer nest, because iterative functions perform many calls and a single return. First-class continuations break call-return nesting even more. However, stack operations (that is, pushes and pops) still nest in these languages, of course. Might and Shivers adapted the call-string mechanism to create frame strings \([7]\), an abstraction that works well for languages with exotic calling behavior.

We already gained some intuition about the use of frame-strings in the last section; stack actions are pushes/pops and they contain the label of the procedure being pushed/popped. We also mark stack actions with timestamps e.g., \(\langle t \text{ square}\rangle\) means popping the frame that holds the arguments of a call to \(\lambda t\text{ square}\) and was first pushed on time \(t\).

A *frame-string* is a sequence of stack actions.

\[
p \in F \iff \langle F \stackrel{\psi}{\leftarrow} | F \stackrel{\psi}{\to} \rangle
\]

* Without \(\text{call/cc}\) this is just popping, with \(\text{call/cc}\) it might also include pushing some frames.

* First-class continuations allow the same frame to be pushed more than once.

Let’s return to our example and see how the stack looks after we push \(\langle 2 \rangle\). Since the frames for \(\text{number?}\) and \(\%\text{If}\) have been popped, the stack is \(\langle\text{square}\rangle \langle 1 \rangle \langle 2 \rangle\). So, by repeatedly cancelling adjacent push/pop actions for the same frame, we get a picture of the stack. We call this *netting* the frame-string:

\[
[p] = \begin{cases} p_1 p_2 & \text{if } p_1(p_1(p_1(\psi^\dagger_p) p_1)) p_2 \lor (p_1 p_1(\psi^\dagger_p)) p_2 \\ \text{otherwise} & \end{cases}
\]

In our example, if we net the frame string that starts with \(\langle 2 \rangle\) and ends with \(\langle\text{halt}\rangle\) we get \(\langle 2 \rangle\langle\text{square}\rangle\langle\text{halt}\rangle\). This gives us the change to the stack after \(\langle 2 \rangle\).

The associative operator + concatenates two frame-strings. Might and Shivers showed that frame-strings modulo netting form a group with respect to concatenation. So, for every frame-string \(p\) there exists another one \(p^{-1}\) such that \(p + p^{-1} = \epsilon\). Intuitively, the inverse string undoes the actions \(p\) did to the stack.

When inverting the concatenation of two frame strings, we know that \((p_1 + p_2)^{-1} = p_2^{-1} + p_1^{-1}\).

To summarize, if the execution of a program is at time \(t\) and we net the frame string from the initial time \(t_0\) to \(t\), we will calculate the stack at time \(t\). Also, if we net the frame string from some past time \(t_0\) to \(t\), we will see the stack change since \(t_0\). Thus, the ability to use frame strings both for recording all stack actions and for finding net stack change makes them a particularly helpful mechanism to reason about the stack.

5. Concrete semantics and stack properties

In this section, we prove that the continuations passed to a user function live on the stack, even in the presence of first-class control (cf. \(\text{Eval}\) case of theorem \([3]\)). To do this, we use the concrete semantics of the \(\Delta\text{CFA}\) analysis \([7]\). This semantics extends \(\kappa\text{-CFA}\) with a log that records frame strings. \(\Delta\text{CFA}\) uses the log only for recording frame strings, not for variable binding or return-point information; these are accomplished using environments, like \(\kappa\text{-CFA}\). The log shows the stack actions that would happen at runtime if the program was compiled using \(\text{ORBIT}\)’s stack policy. Here, we use the log to study the stack behavior of continuations in RCPS.

The semantics and the relevant domains are shown in Fig. \([3]\) at every transition, \(\varsigma\) refers to the state on the left of the arrow. Boldface letters indicate tuples of values. Execution traces alternate between \(\text{Eval}\) and \(\text{Apply}\) states. At an \(\text{Eval}\) state, we use the log to check that the stack is empty at the current call site. At an \(\text{Apply}\) state, we use the log to study the stack behavior of continuations in RCPS.

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mapped to the current time. The new variable environment \(ve'\) maps each \((v_i, t')\) to the corresponding closure \(d_i\).

States use a log \(\delta\) to keep track of the actions they would perform on the stack. We write \(\delta(t)^{-1}\) to mean \((\delta(t))^{-1}\). At each transition from \(\varsigma\) to \(\varsigma'\), \(\rho_\Delta\) records the stack change. To find the stack actions from a time \(t_0\) in the past to \(t'\), we concatenate the actions from \(t_0\) to \(t'\) with \(\rho_\Delta\). Thus, the log \(\delta'\) of \(\varsigma'\) is \((\lambda(t) (\delta(t) + \rho_\Delta))[t' \mapsto \epsilon]\). Naturally, \(\delta'(t)\) is \(\epsilon\) because some time must elapse for stack change to happen.

The stack policy dictates the stack actions \(\rho_\Delta\) at each transition. At \([UEA]\), we must undo all actions that happened since the creation of the youngest continuation argument. We use the function \(youngest\), which takes a set of closures, compares their creation times and returns the most recent time. Then, the stack change should be \(\delta'(youngest(c))^{-1}\). We compute \(\rho_\Delta\) for the other transitions in a similar way. Before calling a continuation, we must reset the stack to the stack of its birth (rule \([CEA]\)). Before entering a function, we push a frame for its arguments (rule \([AE]\)).

We use \(halt\) to denote the top-level continuation of a program \(pr\). The initial state \(I(pr)\) is \((pr, \emptyset, t_0)\), \((halt, \emptyset, [t_0 \mapsto \epsilon], t_0)\).

With the formal machinery in place, we can now show that in a \(UEval\) state \(c\), the frames that make up the environments for the continuation arguments \(q_1, \ldots, q_m\) are still on the stack. When a continuation \(q_j\) is born, its environment is on the top of the stack, so it suffices to show that the net stack change from \(q_j\)'s birth to \(c\) is push-monotonic, meaning a frame string that contains just pushes. (We write \(F\) for the set of push-monotonic frame strings.) In this case, the stack adjustment \(\delta(youngest(c))^{-1}\) in \([UEA]\) transitions consists of series of pops.

By observing the CPS translation of \(call/cc\) you can see why our claim holds even when we allow first-class control: when a continuation is captured by a \(lam\), it can only be called later on, it cannot be passed as an argument to another \(lam\).

To prove push-monotonicity, we will show that each state satisfies a tighter set of constraints (cf. theorem \[3\]). The first constraint is arguably the most important because it talks about stack properties of any continuation closure in \(ve\). The stack motion between the birth of such a closure and the current state can be arbitrary. The constraint guarantees that when a continuation closure is created, it captures continuations that are still on the stack.

Let's look more closely at the creation of continuation-closures.

For every continuation lambda \(\lambda_1\), there is an innermost user lambda \(\lambda_\text{M}\) that contains it. Because of RCPS, \(\lambda_\text{M}\) can only refer to continuation variables bound by \(\lambda_1\). To create a closure \(c\) over \(\lambda_1\), we must first call \(\lambda_1\). Assume that at the time of the call we pass continuations \(c_1, \ldots, c_m\) that are still on the stack. Then, if the net stack motion \(p\) from the call to \(\lambda_1\) to the creation of \(c\) is push-monotonic, \(c_1, \ldots, c_m\) will still be on the stack when \(c\) is created. There are two cases for \(\lambda_\text{M}\); it can appear directly under \(\lambda_1\), e.g., \((\lambda_1(u \ k1 \ k2) (u 15 (\lambda_\text{M} (\text{res}) (+ 4 \text{res} k1)))\)) or after a series of \(C\)Evals whose operators are lambdas, e.g.,

\[\lambda_1(k1) (\langle \lambda_{t1} (u1) \rangle \langle \lambda_{t2} (u2) \rangle (\lambda_\text{M} (\text{res}) (+ 4 \text{res} k1)))\]

In both cases, \(p\) is push-monotonic.

**Definition 2 (Continuation ordering).**

\[\text{Ord}([\lambda(t) (\lambda x (x \varphi))], \beta, ve, \delta, t) \text{ is true iff:}\]

- Let \(k \in \{k_1, \ldots, k_n\}\) and \(ve(k, \beta(k)) = (\text{clam}, \beta', t')\).
  Then, we have that \([\delta(t) + \delta(t)^{-1}] \in F\).
- Let \(k_1, k_2 \in \{k_1, \ldots, k_n\}\),
  \(ve(k_1, \beta(k_1)) = (\text{clam}, \beta_1, t_1)\),
  \(ve(k_2, \beta(k_2)) = (\text{clam}, \beta_2, t_2)\) and \(t_1 \preceq t_2\).
  Then, we have that \([\delta(t_1) + \delta(t_2)^{-1}] \in F\).

**Theorem 3.** Let \(\varsigma\) be a state of the form \((\ldots, ve, \delta, t)\).

For every continuation closure \((\lambda(t) (\text{clam}, \beta, t') \in \text{range}(ve)\),
we have \(\text{Ord}(iu_\text{M}(\text{clam}), \beta, ve, \delta, t)\).

Moreover, depending on the kind of state, we have:

- If \(c \in \text{Eval}, (\lambda(t) (\text{clam}, \beta, ve, \delta, t) \text{ then Ord}(i\text{u}_\text{M}(\text{call}), \beta, ve, \delta, t)\).
- If \(c \in \text{UApply}, ((\lambda(t) (\text{clam}, \beta, ve, \delta, t) \text{ and } c_i = (\text{clam}, \beta, t_i) \text{ then Ord}(i\text{u}_\text{M}(\text{clam}), \beta, ve, \delta, t_i)\text{ and }[\delta(t_i)] \in F\text{ and for each }t_0, t_0 \in \{t_1, \ldots, t_n\} \text{ such that } t_0 \preceq t_i \text{ we have that }[\delta(t_0) + \delta(t_0)^{-1}] \in F\).
- If \(c \in \text{CApply}, ((\lambda(t) (\text{clam}, \beta, t') \text{, \(\beta, ve, \delta, t) \text{ then Ord}(i\text{u}_\text{M}(\text{clam}), \beta, ve, \delta, t)\})

We include the proof of the theorem in the appendix. Note that in a \(CEval\) state, if the operator is a variable but it is not in \(FP(i\text{u}_\text{M}(\text{call}))\), then \(Ord(i\text{u}_\text{M}(\text{call}), \beta, ve, \delta, t)\) guarantees nothing about it; it may be bound to a continuation that has escaped. Therefore, its environment may be popped.

However, in a program without first-class control we can guarantee that continuation environments are never popped because user lambdas do not have free references to continuation variables.
6. Continuation-Age analysis

We now know that continuation environments are still on the stack in \(UEval\) states. This means that we never need to push frames to restore environments in \(UEval\). Also, it means that the environments are totally ordered on the stack at run time. Put formally, if \(t_1\) and \(t_2\) are the birthdays of two continuations then either \([\delta(t_1)]\) is a suffix of \([\delta(t_2)]\) or vice versa. So, if \(t_c\) is the birthday of the youngest continuation then \([\delta(t_c)]\) is a suffix of \([\delta(t_e)]\) where \(t_e\) is the birthday of any other continuation.

So far there has not been an analysis that finds the youngest continuation, and one would have to resort to dynamic checks. We present Continuation-Age analysis (abbrev \(Cage\)) analysis that can find the youngest continuation statically in most cases. We first show the workings of the example by then proceed to develop a formal semantics for it. Consider the following snippet of some RCS program \(pr\):

\[
(\lambda (u_1 \ldots u_5 \ k_1 \ k_2 \ k_3) \ldots (u_2 \ k_1 \ k_3) \ (\lambda (u_5) \ \text{call}) \ (\lambda (u_7) \ \text{call'})) \ldots
\]

Assume that we let \(pr\) run and the execution reaches the call site \(l\). We know that \(k_1, k_2, \text{and} k_3\) are bound to closures whose environments are totally ordered, e.g., with \(k_3\) being the youngest and \(k_2\) the oldest. Also, assume that \(u_7\) is bound to a closure over \(\{\lambda (u_2 \ k_1, k_2, k_3) \ \text{call''}\}\). To find the ordering of the environments at \(l\) we first observe that \(k_2\) is not used at the call site, so we do not take it into account. Also, \(\lambda_i\) and \(\lambda_j\) will evaluate to newly born closures, so the ordering after control enters \(l_2\) will be “\(k_3\) and \(k_2\) followed by \(k_2\) followed by \(k_1\)”. Because of RCS, this is the only information we need to keep to decide the order of continuations inside \(\text{call''}\); remember that \(\{\text{FV}(\text{call'')} \cap \text{CVar}\} \subseteq \{k_2, k_3, k_1, k_2\}\). For this reason, we can analyze simply record total orders of \(c\text{vars}\) bound by the same \(ulam\). It can forget which closures these \(c\text{vars}\) are bound to.

6.1 Concrete semantics

The concrete semantics of \(Cage\) and some auxiliary definitions are shown in Fig. 4. To remove elements from lists we use the set-difference operator, with its meaning adapted in the obvious way. We use \(\text{map}(f, \text{lst})\) to apply a function \(f\) to all elements of \(\text{lst}\). The function \(\text{find(elm, \text{lst})}\) finds the 1-based index of \(elm\) in \(\text{lst}\) and \(\text{get(i, \text{lst})}\) returns the element at index \(i\) in \(\text{lst}\). We also lift \(\text{get}\) and \(\text{find}\) to sets of elements/indices respectively.

The semantic domains are the same as \(k\text{-CFA}\) with the addition of two domains to record the ordering of continuations:

\[
\text{ages, tor} \in \text{Tor} = \langle \text{Pow}(\text{CVar}) \rangle^*
\]

\[
\text{ce} \in CE\text{nv} = U\text{Lab} \times T\text{ime} \rightarrow \text{Tor}
\]

We represent a total order as a list of sets of \(c\text{vars}\), rather than just a list of \(c\text{vars}\), because we want to make explicit the case when two closures are born at the same time. In our example, the order will be \(\{\{k_1, k_2\}, \{k_3\}\}\). The continuation environment \(ce\) maps pairs of user-labels and times to total orders. We write \(k \preceq_{tor} k'\) to mean that the index of \(k\) is smaller than or equal to the index of \(k'\) in \(\text{tor}, i.e., k \text{ is younger than } k'\).

\[
k \preceq_{tor} k' \equiv \exists S', S, k \in S \land k' \in S' \land \text{ind}(S, \text{tor}) \leq \text{ind}(S', \text{tor})
\]

In \(UEval\), we gather order information about the \(ulam\) we are in, and use it to compute order information about the \(ulam\) we are about to enter. Since the new bindings in \(ce\) take place in \(U\text{App}\), \(\text{ages}\) serves as the carrier of that information between states.

\[\text{UEA} \quad \langle f \ e^* q_1 \ldots q_m \rangle \beta, \text{ve}, \text{ce}, t\rangle \rightarrow \langle d_0, \text{d ce}, \text{ve}, \text{ce}, \text{ages}, t'\rangle
\]
\[t' = \text{succ}(s)
\]
\[d_0 = A(f, \beta, \text{ve}, t)\]
\[d_i = A(e_i, \beta, \text{ve}, t)
\]
\[\alpha = A(q_j, \beta, \text{ve}, t)
\]
\[\text{tor} = \langle c(e(VL(q_j), \beta(q_j))) \quad 1 \leq j \leq m. q_j \in \text{Var} \}
\]
\[\emptyset
\]
\[\forall 1 \leq j \leq m. q_j \in \text{Lam}
\]
\[\text{rename}(S) = \text{Get}(\text{Ind}(S, q_i, \ldots q_m), LV(I))
\]
\[\text{ages} = \langle \text{rename}(\text{CLam}) :: \rightarrow \text{map}(\text{rename}, \text{tor}) \rangle \{0\}
\]
\[\text{UAE} \quad \langle d_0, \text{d ce}, \text{ve}, \text{ce}, \text{ages}, t\rangle \rightarrow \langle \text{proc}, \beta, \text{ve'}, \text{ce'}, t'\rangle
\]
\[t' = \text{succ}(s)
\]
\[\beta' = \beta[\text{v} \mapsto \text{t'}]
\]
\[\text{ve'} = \text{ve}[\text{t'}, \text{v'}] \quad \text{ind}(\text{elm}, \text{lst}) = \left\{\begin{array}{ll}
i & \text{lst} = (e_1, \ldots, e_m), \text{elm} = e_i \\
\perp & \text{otherwise}
\end{array}\right.
\]
\[\text{inds}(S, \text{lst}) = \{\text{ind}(s, \text{lst}) | s \in S \setminus \{\perp\}\}
\]
\[\text{get}(i, \text{lst}) = \left\{\begin{array}{ll}
e_i & \text{lst} = (e_1, \ldots, e_m), 1 \leq i \leq m \\
\perp & \text{otherwise}
\end{array}\right.
\]

\[\text{Get}(I, \text{lst}) = \{\text{get}(i, \text{lst}) | i \in I\} \setminus \{\perp\}
\]

Figure 4. Concrete semantics of Cage Analysis

Let’s see how to find the order for the next \(ulam\) using the order of the current \(ulam\). If there are any lambdas among \(q_1 \ldots q_m\), the variables they will be bound to will be the youngest. So \(\text{rename}(\text{CLam})\) gathers the indices of lambdas among \(q_1 \ldots q_m\), and uses them to index in the list of formals of \(\lambda_i\). If every \(q_i\) is a variable, \(\text{rename}(\text{CLam})\) returns the empty set. If there are variables among \(q_1 \ldots q_m\), they are bound by the same \(ulam\), and \(c(e(VL(q_i), \beta(q_i)))\) gathers the order information for that \(ulam\). Then, we filter out variables that are not among \(q_1 \ldots q_m\) and index the rest in the list of formals of \(\lambda_i\). In our example, \(c(e(VL(k_3), \beta(k_3)))\) returns \(\{k_3, k_1, k_2\}\) and \(\text{map}(\text{rename}, \{k_3, k_1, k_2\})\) returns \(\{k_3, k_1, k_2\}\). We remove possible empty sets from our list and we have the new \(\text{ages}\).

Since only user states can influence the ordering, the semantics for \(CE\text{val}\) and \(CA\text{pply}\) are the same as \(k\text{-CFA}\). Note that we can compute continuation ages without using information about the stack actions, thus we do not need a log in the \(Cage\) semantics.

6.2 Abstract semantics

Abstracting the semantics of \(Cage\) raises no difficulty. Like \(k\text{-CFA}\), making the set \(T\text{ime}\) finite ensures computability of the abstract state-space. The abstract counterparts of \(\text{Tor}\) and \(CE\text{nv}\) are
Since one abstract state corresponds to many concrete states, we have to fold many total orders to one element of $\mathbb{T}or$. Thus, the elements of $\mathbb{T}or$ are sets of total orders, with set-union being the
+1
operation. For a $cvar$ to be the youngest in $\mathbb{T}ors$, it has to be the youngest in every total order contained in $\mathbb{T}ors$. This happens because different elements of $\mathbb{T}ors$ correspond to different flows of control in the abstract semantics. Some of these flows may have occurred due to imprecision introduced by the static analysis, but most of them will have a concrete counterpart, so we make sure that all concrete flows agree on the age of $cvars$. We also define maps from the concrete to the abstract domains.

$$
|\mathbb{T}or| = \{\mathbb{T}or\}
|\mathbb{C}e| = (\lambda(l t) \cup \{\mathbb{C}e(l), t\})
$$

The abstract semantics is shown in Fig. 5. Contrary to the concrete semantics, it is non-deterministic. Also, when we add new elements to $\mathbb{C}e$ and $\mathbb{C}e$ we join them instead of doing a destructive update. The two semantics are otherwise similar.

6.3 Soundness

There are two results we need to establish for our analysis to be sound. We first show that a total order of $cvars$ “agrees” with the birthdays of the closures to which these variables are bound.

**Theorem 4.** Let $\xi$ be any state of the form $(\ldots, v_1, v_0, \ldots)$ and $\mathbb{C}e(l, t) = \mathbb{T}or$. If $k_1 \preceq_{\mathbb{T}or} k_2$ and $\mathbb{C}e(k_1, t) = (\mathbb{C}lamb, \beta_1, t_1)$ and $\mathbb{C}e(k_2, t) = (\mathbb{C}lamb, \beta_2, t_2)$ then $t_2 \preceq_{\mathbb{T}or} t_1; \mathbb{C}e(k_1, t)$ was born later than $\mathbb{C}e(k_2, t)$.

Secondly, we show that the abstract semantics simulates the concrete semantics, which means that our approximation is safe.

**Theorem 5 (Soundness of Cage analysis).** If $|\xi| \sqsubseteq \xi$ and $\xi \rightarrow \xi'$ then there exists $\xi''$ such that $\xi \sim \xi''$ and $|\xi''| \sqsubseteq |\xi'|$.

Regarding the time complexity of Cage: since $n$ elements can be totally ordered in $n!$ ways, and the range of $\mathbb{C}Env$ records sets of total orders, the analysis is exponential in max-len-\in \mathbb{U}Q\mathbb{A} \mathbb{V} \mathbb{U} \mathbb{N} (l)$. This is not a problem in practice, since the number of continuation arguments is usually small. A factor that can influence the speed of Cage more dramatically is the choice of $\mathbb{T}ime$ for age analysis, since for $k$ greater than zero $k$-CFA is exponential in the size of the program [13].

Alternatively, there is a less precise lattice we can use for $\mathbb{C}Env$. $\mathbb{C}Env$ can record partial orders of $cvars$ and the join would be set-intersection. Then, $k_1$ would be younger than $k_2$ if $k_1 \sqsubseteq \mathbb{C}e(k_1, k_2)$ and $(k_1, k_2) \in \mathbb{C}e(k_2)$. However, $\mathbb{C}e(k_2)$ introduces more approximation than we would like. For example, consider $\mathbb{C}e_1 = [(k_1, k_2), (k_1, k_3), (k_1, k_4)]$ and $\mathbb{C}e_2 = [(k_1, k_2), (k_1, k_3), (k_1, k_4)]$. Then, $\mathbb{C}e_1 \sqsubseteq \mathbb{C}e_2$ is $0$ even though we know that $k_2$ is never the youngest. In other words, this representation cannot express properties like “either $k_1$ or $k_3$ is younger than $k_2$.”

6.4 Cage vs CFA for age analysis

Theoretically, we could use CFA to find the youngest continuation. Since CFA tracks stack change, we would check if the change between the birthdays of two closures is push monotonic. In practice, this does not work well for the following reasons.

First, variables in the abstract are bound to sets of closures. So, if we want to compare the age of two $cvars$ at a call site, we must check that every closure in one set is younger than every closure in the other set. But then we would end up comparing closures from different flows, which causes imprecision. Cage decouples variables from their bindings and remembers distinct flows as distinct total orders, thus avoiding these problems.

Second, the stack information CFA computes in the abstract can be imprecise in the presence of recursion. It does roughly the following: it can remember exactly one push or one pop action for some $\lambda_\omega$, but if we push two (or more) frames for $\lambda_\omega$, CFA will record this as $|\xi| \sqsubseteq |\xi'\rangle$. Therefore, if we enter a recursive procedure and later return, CFA will not net the pushes and pops. Cage does not suffer from this problem because it does not use the stack to compute continuation age.

7. From $\lambda_{MR}$ to RCPS

The multi-return $\lambda$-calculus [12] is a variant of the $\lambda$-calculus in which functions may have many return points. Return points are not first-class continuations, hence they give the programmer the ability to express a wide variety of algorithms without paying the
cost of general-purpose, heap-allocated continuations. Search algorithms that take a success and a failure continuation, functional tree transformations and LR-parsers are typical examples of programs that are naturally and efficiently expressed with this mechanism.

The multi-return form \( \text{\texttt{e r}} \) is how we get contexts with many return points. The expression \( e \) is evaluated with \( m \) return points in scope. If \( e \) does not use the multi-return form internally, it will always return to the first one, as in the first example of Fig. 6. However, if \( e \) is of the form \( \texttt{e'} \# \# \# \) then the result of \( e' \) will be passed to \( r_i \), as in the second example. A return point \( \# i \) passes its input to the \( i^{\text{th}} \) return point of its own context.

Restricted CPS, with the restrictions it places on continuations, would seem like a natural target for \( \lambda_{\text{MR}} \). However, a subtlety of \( \lambda_{\text{MR}} \) is that functions are polymorphic in the number of return points that they expect, they do not specify it explicitly in their syntax. The last snippet of Fig. 6 is one such example. Depending on the result of the test, the square function will be evaluated in a context with one or two return points, even though it always returns to the first. Since in RCS a \( \texttt{ulam} \) has to specify the number of continuations it expects, we cannot translate this code to RCS.

A control-monomorphic variant however has a simple transform to RCS. We require that a function take a specific number of return points, which we pass when we apply the function. We change the syntax and semantics of \( \lambda_{\text{MR}} \) slightly to reflect this (section 7.1). We provide a type system that rejects control-polymorphic \( \lambda_{\text{MR}} \) programs and prove it sound (section 7.2). Then, we give a type-directed transform from \( \lambda_{\text{MR}} \) to RCS (section 7.3).

### 7.1 Syntax and semantics

Expressions in control-monomorphic \( \lambda_{\text{MR}} \) include variables, numbers, functions, applications with a specified number of return points and multi-return forms. Numbers and variables are values:

\[
\begin{align*}
\text{\texttt{lam}} & \in \text{Lam} ::= (\lambda(x) \ e) \\
\text{\texttt{e}} & \in \text{Exp ::= \{lam | e \}_{1 \ldots m}} \\
\text{\texttt{r}} & \in \text{RP ::= \{lam \}_{1 \ldots m}}
\end{align*}
\]

The semantics is call-by-value (Fig. 7). To evaluate \( \texttt{e r} \) we first evaluate \( e \) in a context with \( m \) return points (multi-prog). If it reduces to a value \( v \) and there is a single return point which is a function, we apply it to \( v \) (lst-fam). If the single return point is \( \# 1 \) we return \( v \) to the context (lst-sharp). When there are multiple return points, \( v \) is returned to the first one (multi-drop). If \( e \) evaluates to \( \# u \) \( \# b \) in a context with \( i \) more return points then we pass \( v \) to \( r_i \) (multi-select).

In an application we start with the operator (rator), then the operand (rand) and then the body of the function (app). These rules highlight the difference from control-polymorphic \( \lambda_{\text{MR}} \). Unlike the last example of Fig. 6, we have to mention the return points when applying a function. Our type system checks that a function is always applied in contexts with the same number of return points.

A note about the stack behavior of \( \lambda_{\text{MR}} \) deserves a mention. When a return point is a function, it requires a frame to be pushed, while a \( \# i \) return point just points to an older frame. Thus, when all return points of \( \text{\texttt{e r}} \) are not functions, the stack does not grow, and it might even shrink. This is essentially the tail call mechanism applied to \( \lambda_{\text{MR}} \).

### 7.2 Types for control-monomorphism

We modify the original type system of \( \lambda_{\text{MR}} \) to annotate function types with the number of return points that a function expects.

Each expression \( e \) is assigned a type vector \( \langle \tau_1, \ldots, \tau_m \rangle \) meaning that if \( e \) returns a value \( v \) to its \( i^{\text{th}} \) return point, \( v \) has type \( \tau_i \).

Placing \( \bot \) instead of a type at index \( i \) means that \( e \) never returns to that return point. For example, \( \langle 11 \# 2 \rangle \) has type \( (\bot, \bot) \). But \( \langle 11 \# 2 \rangle \) never returns to any return point \( r_i \) for \( i > 2 \). Hence it can also have type \( (\bot, \bot, \bot) \), \( (\bot, \bot, \bot, \bot) \), etc. Moreover, (int, int) is also a possible type since the requirement "if \( \langle 11 \# 2 \rangle \) returns to its first return point it gives back an integer" is vacuously true. To model these, our type system has a notion of subtyping.

Types include integers and functions, and type vectors \( \tau \) are finite maps from natural numbers to types. Then, \( \tau[i] = \bot \) means that \( i \notin \text{dom}(\tau) \).

\[
\begin{align*}
\tau & \in T ::= \text{int} \{ (\tau, n) \to \tau \} \\
\tau & \in T' = \ N \to T
\end{align*}
\]

Function types include a natural number \( n \), meaning that \( n \) return points must be provided when a function \( f \) is applied. Obviously, we run into trouble if \( f \) tries to return to \( r_i \) for \( i > n \). Therefore, we require that \( \tau[i] \leq n \) where \( \tau[i] \) is \( \text{true} \) for \( i \leq j \) and \( \text{false} \) otherwise.

The subtyping rules for types and vectors are

\[
\begin{align*}
\text{int} & \subseteq \text{int} \{ (\tau, n) \to \tau \} \\
(\tau, n) & \to \tau_i \subseteq (\tau, n) \to \tau_i
\end{align*}
\]

\[
\forall i \in \text{dom}(\tau), \ i \in \text{dom}(\tau_i) \land \tau_i[i] \subseteq \tau_i[i] \subseteq \tau_i \subseteq \tau
\]

The type system is shown in Fig. 8. It assigns type vectors to expressions under an environment \( \Gamma \) which is a partial map from variables to types.

The rules for numbers and variables are standard (num, var).

To typecheck a function \( (\lambda(x) \ e) \), we typecheck its body in an environment extended with \( x \). The side condition states that if the function uses \( \tau[i] \) return points then it must request at least as many in its type.

For an application \( \langle e_1 e_2 \rangle \ r_1 \ldots r_m \) we require that \( e_1 \) have a function type with exactly \( m \) return points (appl). The type of the argument must be a subtype of what the function expects (side condition 1). If the \( j^{\text{th}} \) return point is a \( \text{\texttt{lam}} \) with type \( (\tau_j, m_j) \to \tau_j \), then anything that \( e_1 \) returns to it must be a subtype of \( \tau_j \). Additionally, anything that \( r_j \) returns to the context must be consistent with what the whole expression returns. For this, we require \( \tau_i \subseteq \tau_{\text{app}} \) (side condition 2). On the other hand, if the return point is of the form \( \# i \) then whatever \( e_1 \) returns to its \( j^{\text{th}} \) return point will be sent to the context’s \( i^{\text{th}} \) return point, which is why we require \( \tau[j] \subseteq \tau_{\text{app}} \) (side condition 3).

For a \( r_1 \ldots r_m \) expression (multi) the typing constraints required from return points are the same as in the application case (side conditions 2, 3). We also require that \( e \) only try to return to \( r_1 \ldots r_m \) (side condition 1).

We can now see why the type system rejects control-polymorphic \( \lambda_{\text{MR}} \) programs. The operator of our last example is

\[
(\lambda(x) \text{ if test } \langle f \rangle 2 \ (\lambda(x) \ x + 1) (\lambda(x) \ x - 1) \rangle
\]

The true-branch requires \( f \) to have a type of the form \( (\text{int, int}) \to \tau \) and the false-branch requires \( f \) to have a type of the form \( (\text{int, int, int, int}) \to \tau \). Since none of these types is a subtype of the other, the body cannot be typechecked with a unique type for \( f \).

We split the type-soundness proof in the progress and preservation theorems.

**Theorem 6 (Progress).**

If \( \Gamma \vdash e : \tau \) and \( e \) is closed then either \( e \) is a value, or \( e \) is of the form \( \langle \_ \# b \rangle \) where \( i > 1 \), or \( e \to e' \).
---

**Figure 7.** Operational Semantics of λ_{MR}

### [num]
\[
\Gamma \vdash n : \langle \text{int} \rangle \\
\Gamma \vdash x : \langle \Gamma(x) \rangle \quad x \in \text{dom}(\Gamma)
\]

### [var]
\[
\Gamma \vdash \lambda : \langle \text{int} \rangle \quad x \in \text{dom}(\Gamma)
\]

### [abs]
\[
\Gamma[x \mapsto \tau] \vdash e : \forall \quad n \geq |\tau|
\]

### [appl]
\[
\Gamma \vdash e_1 : \langle (\tau, m) \rightarrow \tau \rangle \\
\Gamma \vdash e_2 : \tau_2 \\
\Gamma \vdash r_j : \langle (\tau_j, m_j) \rightarrow \tau_j \rangle (\forall r_j \in \text{Lam}) \\
\Gamma \vdash \lambda (e_1 e_2) : \tau_{app} \\
\]

### [multi]
\[
\Gamma \vdash e : \tau_i \\
\Gamma \vdash r_j : \langle (\tau_j, m_j) \rightarrow \tau_j \rangle (\forall r_j \in \text{Lam}) \\
\Gamma \vdash \lambda (e r_j) : \tau
\]

**Trivial Term:**
\[
\forall e. n(x) = x \\
\forall e. [n] = n \\
\forall e. \lambda (x e) = \lambda (x k_1 \ldots k_m)
\]

**Return Point:**
\[
\forall e. [\#] k_1 \ldots k_i = k_i \\
\forall e. \lambda (x e) k_1 \ldots k_i = \lambda (x) e k_1 \ldots k_i
\]

** Serious Term:**
\[
\forall e. [\#] k_1 \ldots k_i = (k_1 \text{\#} [\#] k_0) \\
\begin{align*}
\forall e. [\#] k_1 \ldots k_i & = (\forall k_i \text{ is a variable,} \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,})
\end{align*}
\]

**If every k_i is a variable,**
\[
\begin{align*}
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,})
\end{align*}
\]

**If every k_i is a variable,**
\[
\begin{align*}
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,})
\end{align*}
\]

**If there exists a \text{lam} among k_1 \ldots k_i,**
\[
\begin{align*}
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,})
\end{align*}
\]

**If there exists a \text{lam} among k_1 \ldots k_i,**
\[
\begin{align*}
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,})
\end{align*}
\]

**If there exists a \text{lam} among k_1 \ldots k_i,**
\[
\begin{align*}
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,}) \\
& \forall e. [\#] [\#] k_1 \ldots k_i = (\forall k_i \text{ is a variable,})
\end{align*}
\]

**Transformation of \lambda_{MR} to Restricted CPS**

---
Theorem 7 (Preservation).
If $\Gamma \vdash e : \tau$ and $e \mapsto e'$ then $\Gamma \vdash e' : \tau'$ where $\tau' \subseteq \tau$.

Both proofs proceed by structural induction on e. In the progress theorem, note that a well-typed expression does not always reduce to a value. It might evaluate to a multi-return form that cannot take any steps. The proofs require the following lemmas.

Lemma 8 (Weakening).
If $\Gamma[x \mapsto \tau]$ $\vdash e : \bar{\tau}$ and $x \notin FV(e)$ then $\Gamma$ $\vdash e : \bar{\tau}$.

Lemma 9 (Substitution).
If $\Gamma[x \mapsto \tau]$ $\vdash e : \tau_1$, $e'$ is closed and has type $\tau_1 \mapsto \tau'$, and $\tau' \subseteq \tau_2$ such that $\tau_2 \subseteq \tau_1$.

7.3 Transformation of $\lambda_{MR}$ to RCPS

In this section, we describe a CPS transformation from $\lambda_{MR}$ to RCPS (Fig. 7). Fisher and Shivers have shown that multi-return functions are cheap to implement and do not require novel compilation techniques. By translating $\lambda_{MR}$ to RCPS, it becomes amenable to Cage Analysis which can further improve performance.

The transform relies on information provided by the type system to add the correct number of continuation parameters to ulams. We use standard techniques [3] to make the transform compositional and first-order. Last, some effort is spent on making sure that the transform does not duplicate code.

The transform uses three mutually recursive functions, for trivial terms, serious terms and return points. Variables and values are trivial terms and the rest are serious. The metavariables $t$ and $s$ range over trivial and serious terms respectively. Underlined lambdas $\lambda$ generate fresh identifiers to avoid variable capture. We apply the transform to a $\lambda_{MR}$ program $e$ by calling $\mathcal{S}[e]halt$.

The translation of variables and numbers is straightforward. When translating a $\lambda_{MR}$ term we look at its type to find out how many continuations it takes in CPS.

A $\#_1$ return point becomes a reference to the $i^{th}$ continuation of its context. A $(\lambda(x)e)$ return point becomes a $clam$ in CPS. Here, there is possible code duplication that we want to avoid. Assume that one of $k_1 \ldots k_n$ is a $clam$. Then, if $e$ refers to the corresponding return point more than once, this $clam$ will be duplicated. For this reason, the rest of the rules call $\mathcal{S}$ with $cvars$ arguments only.

If $\mathcal{S}$ is applied to a trivial term then we return the term to the first continuation. Application is split in four cases depending on the operator and the operand. Note how the continuations $k_1 \ldots k_n$ are passed to all return points, which is why we require that they all be $cvars$ to prevent duplication. If there is a $clam$ among $k_1 \ldots k_n$ we create a new $ulam$ and transform the application using the new $cvars$.

For $\mathcal{S} e \tau_1 \ldots \tau_m$, we have to translate $e$ in a context with $m$ continuations. Here again we split in two cases to avoid duplication.

It is simple to see why our transformation generates RCPS code. The only place where a $ulam$ is generated is the rule $\mathcal{S}[(\lambda(x)e)]$, and we pass only the newly-created $cvars$ to $e$.

The duplication of code is best seen in an example. Assume that we omit the rules that take care of $clams$ in $k_1 \ldots k_n$. Then, all continuation arguments are treated the way variables are now treated. In the following transform, the return point $(\lambda(y)e')$ will be duplicated in the RCPS output:

\[ \mathcal{S}[\langle\langle (\lambda(x)e) 42 \rangle \rangle] #1 \#1 \triangleright (\lambda(y)e')halt \]
\[ = \mathcal{S}[\langle\langle (\lambda(x)e) 42 \rangle \rangle] #1 \#1 \triangleright (\lambda(y)\mathcal{S}[\langle\langle e \rangle \rangle]halt) \]
\[ = (\lambda(x)k_1k_2)\mathcal{S}[\langle\langle e \rangle \rangle]k_1k_2 42 \]
\[ (\lambda(y)\mathcal{S}[\langle\langle e \rangle \rangle]halt) \]
\[ (\lambda(y)\mathcal{S}[\langle\langle e \rangle \rangle]halt) \]

On the other hand, our transformation yields the more compact:
\[ ((\lambda(k)(\lambda(x)k_1k_2)\mathcal{S}[\langle\langle e \rangle \rangle]k_1k_2 42 k k)) (\lambda(y)\mathcal{S}[\langle\langle e \rangle \rangle]halt)) \]

8. Evaluation of Cage

We implemented Cage in Scheme48. Our compiler takes a multi-return Scheme program to RCPS, on which it runs Cage. It does not go all the way down to machine code. We measured the precision by counting the multiple-continuation call sites for which the analysis can find the youngest continuation statically. The results are encouraging, since the analysis is very precise, with little additional cost in running time and implementation effort over k-CFA.

Our analysis handles a purely functional subset of Scheme with numbers, booleans, lists, explicit recursion, and multi-return functions. We changed the front end of Scheme48 to accept a multi-return construct. After the front end takes care of parsing and macro-expansion, every call in the AST is represented as a multi-return call, e.g., (+ 1 2) becomes ($\langle\langle + (1 2) \rangle \rangle \#1 \#1$). This makes the conversion to RCPS more uniform. The compiler then runs Cage, followed by a final linear pass that computes the results per call site (since ages in $\tilde{\epsilon}$ are grouped by $ulam$ labels). For instance, assume that, for the lambda expression $(\lambda(x)(\#1 \#1 k_2) (\#1 (1 2 3) k_1))$, Cage finds that $k_1$ is younger than $k_2$ in every total order container in $\bigcup_{l \in \gamma_{MRR}} \tilde{\epsilon}(l, i)$. Then, the final pass will deduce that $k_1$ is always younger than $k_2$ at $\gamma$. Our current implementation spots the opportunity for optimization and stops. However, this information could be passed to a code-generation phase, which would avoid emitting code to check the ages of continuations at $\gamma$.

Fisher and Shivers suggested that LR-parsers can be compiled to $\lambda_{MR}$ with considerable speed gains [12]. Each state of the parser’s automaton is represented as a function; a shift is a function call. Reductions do not return to a state function’s immediate caller; but to points higher in the stack. This is handled with multiple return points to point to the necessary frames; a simple analysis determines how these return points represent the target reduction states.

Such parsers contain an abundance of multi-continuation calls, which makes them attractive benchmarks for Cage. We ran Cage (with $k = 0$) on a parser for a medium-sized, Pascal-like language. Out of the 973 calls to $ulams$, 152 pass two continuations and 32 pass three. If there is a $clam$ argument, it is trivially the youngest continuation. This happens in 20 calls. The remaining 164 pass only $cvars$. Cage found the youngest continuation in 142, and in 22 calls it narrowed the youngest down to two choices instead of three. There was no call site for which the analysis failed to gain at least partial information. Cage amounts to 19.8% of the total running time of the abstract interpretation (the rest is spent on flow analysis), and 32.2% of the code size.

The effectiveness of the analysis is also illustrated by tail-recursive programs that can throw exceptions. The RCPS program of Fig. 10 sums all numbers in a list 1 and returns to ee, or throws an exception by calling h if it finds a non-number in 1. It could have been written originally in any language with exceptions, or in a multi-return language. Placed in some code that computes the sum of a list of lists of numbers, this essentially becomes the inner loop, so optimizing it is crucial. Cage statically figures out that the continuations in the recursive call have the same age.

In the following program, Cage fails to figure out the youngest continuation passed to $\lambda_{\tilde{S}}$ when $k$ is 0. That is because in $l/2$ the first

\[ \mathcal{S}[\langle\langle (\lambda(x)e) 42 \rangle \rangle] #1 \#1 \triangleright (\lambda(y)e')halt \]
\[ = \mathcal{S}[\langle\langle (\lambda(x)e) 42 \rangle \rangle] #1 \#1 \triangleright (\lambda(y)\mathcal{S}[\langle\langle e \rangle \rangle]halt) \]
\[ = (\lambda(x)k_1k_2)\mathcal{S}[\langle\langle e \rangle \rangle]k_1k_2 42 \]
\[ (\lambda(y)\mathcal{S}[\langle\langle e \rangle \rangle]halt) \]
\[ (\lambda(y)\mathcal{S}[\langle\langle e \rangle \rangle]halt) \]
formations. We use several elements of
been primarily used to show environment equivalence and related
functional languages with first-class control. To date, \( \Delta \)CFA is a static analysis that can reason about stack change in

\[
\text{Cage} \quad \lambda (\text{test2}) \\
(\text{if} \text{test2}) \quad (\lambda x) \\
(\text{cdr} l) \quad (\lambda (\text{rest}) \\
(+ \text{acc} \text{fst}) \\
(\lambda (\text{sum}) (\text{suml} \text{rest sum cc h]))) \\
(\lambda (\text{h} \text{not a number})))
\]

**Figure 10.** Tail recursion with exceptions

continuation is the youngest, and in \( l3 \) the second. Similar examples can be written for any \( k \):

\[
((\lambda (f) (\text{if some-test}) \\
\quad (\lambda (x) (k x)) k^{1/2}) \\
\quad (\lambda (f k (\lambda (y) (k y)) y^{1/2}))) \\
\quad (\lambda (k1 k2) \ldots )^{1/3} \\
\quad \text{halt}
\]

Overall, we are satisfied with the precision of Cage. It remains
to be seen how useful it is in practice. More experience with multi-
return code and multi-continuation CPS is needed to see if execu-
only call sites show up as often as in the programs presented here.

9. Related work

CPS was first formalized by Plotkin [8] and was used as an IR
in Rabbit [14] and ORBIT [5], which were early and influential
compilers for Scheme. Shivers used CPS to solve the control-flow

The starting point for the present work has been \( \Delta \)CFA [6,7].
\( \Delta \)CFA is a static analysis that can reason about stack change in
functional languages with first-class control. To date, \( \Delta \)CFA has
been primarily used to show environment equivalence and related
optimizations, but it enables, in principle, many stack-related trans-
formations. We use several elements of \( \Delta \)CFA in this paper. First,
we base our Restricted CPS on Partitioned CPS. More important-
ly, we use frame strings and the concrete semantics of \( \Delta \)CFA to prove
that continuation arguments of ulams are still on the stack.

Kennedy [4] proposed a variant of CPS which, like ORBIT, pro-
vides a variety of choices for procedures. He argues that CPS is
preferable over ANF and monadic languages because function in-
lining does not require renormalization steps or the use of commut-
ing conversions. Also, he advocates CPS as a suitable IR even in
the absence of first-class control in the source language. Kennedy’s
CPS satisfies some syntactic restrictions similar to Restricted CPS.

The main differences are that his CPS does not deal with first-class
control and that user lambda can take up to two continuation argu-
ments, the current continuation and a handler continuation. If a
ulam can throw many exceptions, the handler must be polymor-
phic; in RCPS we can pass as many continuations as needed.

There has been significant work done on efficient run-time im-
plementations of first-class continuations, that is, continuations that
outlive their dynamic extent and so require the stack to be saved in
the heap [2]. Our work here, however, focuses on demonstrating
the circumstances under which we may safely assume that continu-
ations need not be copied, and on reasoning about the relationships
between different continuations that are known to live on the stack.

10. Conclusions

In this paper, we show how a simple syntactic constraint on a CPS
intermediate representation enables efficient use of the stack in the
presence of multiple continuations. We prove that when we pass
many continuations to a user function their environments are still
on the stack. The generalization of the tail-call mechanism dictates
that we pop to the most recent of these frames before control enters
a user function.

We proceed to develop Cage, an analysis that finds the youngest
frame at compile time in most cases. The main idea behind Cage
is that inside a function \( \{\lambda (\delta1 \ldots \delta n) k1 \ldots k_m \text{ calM]\} \) we only
need to remember age information about \( \delta_1 \ldots \delta_n \), we can forget
which closures these variables are bound to. This decoupling be-
tween variables and bindings is possible because of Restricted CPS.

A prototype implementation of Cage in Scheme48 shows that
it is a precise analysis with little extra overhead in compilation
time over \( k \)-CFA. Therefore, control constructs that require passing
many continuations, like exceptions and multi-return functions, can
be compiled to fast native code.

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A.

Lemma 10. For each state with \( \log \delta \) and time \( t \), \( \delta(t) = \varepsilon \).

Proof. By looking at the transition rules, it is immediately obvious that the lemma holds for \( I(pr) \) and is maintained by transition. \( \square \)

Lemma 11. Let \( \text{Ord}(\text{lam}_b, \beta, \nu, \delta, t) \) and, for some \( k \Delta, \delta' = (\lambda(t)(\delta(t) + p_\Delta))^t' \rightarrow \varepsilon \). Then, \( \text{Ord}(\text{lam}_b, \beta, \nu, \delta', t) \)

Proof. Intuitively, the lemma holds because the stack actions that happened after time \( t \) do not matter.

Let the set \( S \) be \( FP(\text{lam}_b) \cup CFam \).

To prove \( \text{Ord}(\text{lam}_b, \beta, \nu, \delta', t) \), we have two obigations.

- Let \( k \in S \) and \( \nu(k, \beta(k)) = (\text{clam}_k, \beta_k, t_k) \). We must show that \( [\delta'(t_k) + \delta'(t')^{-1}] \in \bar{F} \)
  \( \Leftrightarrow [\delta(t_k) + p_\Delta + (\delta(t) + p_\Delta)^{-1}] \in \bar{F} \)
  \( \Leftrightarrow [\delta(t_k) + p_\Delta + p_\Delta^{-1} + \delta(t')^{-1}] \in \bar{F} \)
  \( \Leftrightarrow [\delta(t_k) + \delta'(t)^{-1}] \in \bar{F} \)
  \( \Leftrightarrow \text{Ord}(\text{lam}_b, \beta, \nu, \delta, t) \)

- Second, let \( k_1, k_2 \in S \), \( \nu(k_1, \beta(k_1)) = (\text{clam}_1, \beta_1, t_1) \), \( \nu(k_2, \beta(k_2)) = (\text{clam}_2, \beta_2, t_2) \), and \( t_1 \leq t_2 \). We must show that \( [\delta'(t_1) + \delta'(t)^{-1}] \in \bar{F} \)
  \( \Leftrightarrow [\delta(t_1) + p_\Delta + p_\Delta^{-1} + \delta(t')^{-1}] \in \bar{F} \)
  \( \Leftrightarrow [\delta(t_1) + \delta(t_2)^{-1}] \in \bar{F} \)
  \( \Leftrightarrow \text{Ord}(\text{lam}_b, \beta, \nu, \delta, t) \)

- Second, for any \( c_i \), we must show \( \delta(t) \in \bar{F} \), i.e.,

\[ [\delta(t) + p_\Delta] \in \bar{F} \]

We know two things about the UEV state.

For every closure \( (\text{clam}_c, \beta_c, t_c) \in \text{range}(\nu) \), we know

\[ \text{Ord}(\text{iu}(\text{clam}_c), \beta_c, \nu, \delta, t_c) \] (1)

Also, we know

\[ \text{Ord}(\text{iu}(\text{lam}_b), \beta, \nu, \delta, t) \] (2)

If one of \( q_1 \ldots q_n \) is a lambda, it will result in a new continuation closure so we get

\[ p_\Delta = \delta(t) \] (3)

First, we show that continuations in \( \nu \) of the UApply obey \( \text{Ord} \).

The variable environment doesn’t change in the transition, so we must show

\[ \text{Ord}(\text{iu}(\text{clam}_c), \beta_c, \nu, \delta^t, t_c) \] (4)

which follows from (1) and lemma (11)

We have three obligations about the UApply state.

- First, for any \( c_i \), we must show

\[ \text{Ord}(\text{iu}(\text{clam}_c), \beta_c, \nu, \delta^t, t_c) \] (5)

There are two options for \( q_i \).
If \( q_i \in \text{CLam} \) then (5) holds if

\[ \text{Ord}(\text{iu}(\text{lam}_b), \beta, \nu, \delta, t) \] (6)

If \( q_i \in \text{CA} \) and \( q_1 \ldots q_m \) are variables, let \( n \) be the index of the youngest continuation. Then, \( (6) \) follows from (4).

- Second, for any \( c_i \), we must show \( \delta(t) \in \bar{F} \), i.e.,

\[ [\delta(t) + p_\Delta] \in \bar{F} \]

If \( q_i \in \text{CLam} \), by (3) it suffices to show \( \nu(\text{value}) \in \bar{F} \), which holds.

If \( q_i \in \text{CA} \) and there is a lambda among \( q_1 \ldots q_m \), then (6) holds if

\[ \text{Ord}(\text{iu}(\text{lam}_b), \beta, \nu, \delta, t) \] (7)

If \( q_i \in \text{CA} \) and all \( q_1 \ldots q_m \) are variables, let \( n \) be the index of the youngest continuation. Then, (6) holds if

\[ \text{Ord}(\text{iu}(\text{lam}_b), \beta, \nu, \delta, t) \] (8)

Theorem. Let \( c \) be a state of the form \( \langle \ldots \nu, \beta, \delta, t \rangle \).

For every continuation closure \( (\text{clam}_c, \beta_c, t_c) \in \text{range}(\nu) \), we have \( \text{Ord}(\text{iu}(\text{clam}_c), \beta_c, \nu, \delta, t_c) \).

Moreover, depending on the kind of state, we have:

- If \( \xi \in \text{Eval} \), \( \langle \nu, \beta, \delta, t \rangle \) then \( \text{Ord}(\text{iu}(\text{clam}_c), \beta_c, \nu, \delta, t) \)
- If \( \xi \in \text{UApply} \), \( \langle \text{lam}_b, \beta, t' \rangle \), \( \xi \equiv \text{clam}_c \ldots \text{clam}_n, \nu, \beta, t \) and \( c_i = (\text{clam}_i, \beta_i, t_i) \) then \( \text{Ord}(\text{iu}(\text{clam}_c), \beta_c, \nu, \delta, t_i) \)

\[ [\delta(t_i)] \in \bar{F} \] and for each \( t_a, t_b \in \{t_1, \ldots, t_n \} \) such that \( t_a \leq t_b \), we have that \( [\delta(t_a) + p_\Delta^{-1}] \in \bar{F} \)

- If \( \xi \in \text{CA} \), \( \langle \text{clam}_c, \beta_c, t_c \rangle, \nu, \beta, \delta, t \) then \( \text{Ord}(\text{iu}(\text{clam}_c), \beta_c, \nu, \delta, t_c) \)

[UAE] The transition is

\[ \langle \lambda(\nu' k_1 \ldots k_m) \rangle \langle \text{clam}_c \ldots \text{clam}_n, \nu, \beta, t \rangle \rightarrow \langle \text{call} \rangle, \nu', \beta', t' \]

\[ \beta' = \beta' \rightarrow t' \]

\[ \nu' \rightarrow \text{clam}_c \]

\[ p_\Delta = \langle \delta(t) + p_\Delta \rangle \] (9)

We are being slightly sloppy with the user arguments because they are not relevant in the proof.

For every closure \( (\text{clam}_c, \beta_c, t_c) \in \text{range}(\nu) \), we know

\[ \text{Ord}(\text{iu}(\text{clam}_c), \beta_c, \nu, \delta, t_c) \] (10)

For each \( c_i \), the form \( (\text{clam}_c, \beta_c, t_c) \), we know

\[ \text{Ord}(\text{iu}(\text{clam}_c), \beta_c, \nu, \delta, t_c) \] (11)
First, we will show that continuations in the variable environment \( v'e \) of the successor state obey \( \text{Ord} \).

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, v'e, \delta', t_c)) \tag{12}
\]

If a continuation in \( v'e \) was already in \( v \), then

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, v'e, \delta, t_c)) \tag{11}
\]

If a continuation in \( v'e \)’s one of \( c_1 \ldots c_n \), then

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, v'e, \delta, t_c)) \tag{12}
\]

Second, we must show

\[
\text{Ord}(\{\lambda_i (u^* k_1 \ldots k_n) \text{ call} \}), \beta', v'e, \delta', t') \tag{13}
\]

The first condition for (13) requires that for each closure \( c_i \) we have

\[
|\delta'(t_i) + \delta'(t_i') - 1| \in F
\]

This holds if

\[
\delta'(t_i) \in F
\]

\[
\delta'(t_i') \notin F
\]

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, \delta', t_c)) \tag{11}
\]

The second condition for (13) requires that for any two closures in \( c \) with birth times \( t_1 \) and \( t_2 \) such that \( t_1 \leq t_2 \), it holds that

\[
|\delta'(t_1) + \delta'(t_2) - 1| \in F
\]

We know that all closures in \( c \) were born before the \( U\text{Apply} \) state, so their birth times are earlier than \( t \), thus

\[
|\delta'(t_1) + \delta'(t_2) - 1| \in F
\]

\[
\delta'(t_1') \in F
\]

\[
\delta'(t_2') \notin F
\]

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, \delta, t_c)) \tag{11}
\]

[CEA]

The transition is

\[
(\{q e\}), \beta, v, \delta, t \rightarrow (\text{proc}, d, v, \delta', t')
\]

\[
\delta' = (\text{succ}(\delta) + p_{\Delta}) \to \lambda (t) \delta(t) + p_{\Delta}) \to |

For every closure \( (\text{clam}, \beta, t_c) \in \text{range}(v) \), we know

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, v, \delta, t_c)) \tag{15}
\]

Also, we know

\[
\text{Ord}(\text{iuc}_k(\{q e\}), \beta, v, \delta, t) \tag{16}
\]

We must show

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, v, \delta, t_c)) \tag{17}
\]

which follows from (15) and lemma (11)

Second, we must show

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, v, \delta, t')) \tag{18}
\]

If \( q \in \text{CLam} \) then

\[
\text{clam} = q, \beta \in \beta, t_r = t, p_{\Delta} = \varepsilon, \delta = \delta[t' \mapsto \varepsilon]
\]

For each continuation variable \( k \in \text{FP}(\text{iuc}_k(q)) \) where \( \text{ve}(k, \beta, k) = (\text{clam}_k, \beta_k, t_k) \), we must show

\[
|\delta'(t_k) + \delta'(t_k') - 1| \in F
\]

\[
\delta'(t_k) \in F
\]

\[
\delta'(t_k') \notin F
\]

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, v, \delta, t')) \tag{16}
\]

For each continuation variables \( k_1, k_2 \in \text{FP}(\text{iuc}_k(q)) \) where \( \text{ve}(k_1, \beta, k_1) = (\text{clam}_1, \beta_1, t_1), \text{ve}(k_2, \beta, k_2) = (\text{clam}_2, \beta_2, t_2) \), we must show

\[
|\delta'(t_1) + \delta'(t_2) - 1| \in F
\]

\[
|\delta'(t_1) + \delta'(t_2) - 1| \in F
\]

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, v, \delta, t')) \tag{16}
\]

For each continuation variable \( k \in \text{FP}(\text{iuc}_k(q)) \) where \( \text{ve}(k, \beta, k) = (\text{clam}_k, \beta_k, t_k) \), it holds that

\[
|\delta'(t_k) + \delta'(t_k') - 1| \in F
\]

\[
\delta'(t_k) \in F
\]

\[
\delta'(t_k') \notin F
\]

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, v, \delta, t_k)) \tag{12}
\]

\[
|\delta'(t_k) + \delta'(t_k') - 1| \in F
\]

\[
\delta'(t_k) \in F
\]

\[
\delta'(t_k') \notin F
\]

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, v, \delta, t_k)) \tag{12}
\]

\[
|\delta'(t_k) + \delta'(t_k') - 1| \in F
\]

\[
\delta'(t_k) \in F
\]

\[
\delta'(t_k') \notin F
\]

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, v, \delta, t_k)) \tag{12}
\]

\[
|\delta'(t_k) + \delta'(t_k') - 1| \in F
\]

\[
\delta'(t_k) \in F
\]

\[
\delta'(t_k') \notin F
\]

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, v, \delta, t_k)) \tag{12}
\]

\[
|\delta'(t_k) + \delta'(t_k') - 1| \in F
\]

\[
\delta'(t_k) \in F
\]

\[
\delta'(t_k') \notin F
\]

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, v, \delta, t_k)) \tag{12}
\]

\[
|\delta'(t_k) + \delta'(t_k') - 1| \in F
\]

\[
\delta'(t_k) \in F
\]

\[
\delta'(t_k') \notin F
\]

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, v, \delta, t_k)) \tag{12}
\]

\[
|\delta'(t_k) + \delta'(t_k') - 1| \in F
\]

\[
\delta'(t_k) \in F
\]

\[
\delta'(t_k') \notin F
\]

\[
\text{Ord}(\text{iuc}_k(\text{clam}, \beta_k, v, \delta, t_k)) \tag{12}
\]