Basic Probability and Decisions

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Uncertainty

Let action A_t = leave for airport *t* minutes before flight
Will A_t get me there on time?

Problems:

1. partial observability (road state, other drivers' plans, etc.)
2. noisy sensors (traffic reports)
3. uncertainty in action outcomes (flat tire, etc.)
4. immense complexity of modeling and predicting traffic

Hence a purely deterministic (logical) approach either
1) risks falsehood: "A_{25} will get me there on time" or 2) leads to conclusions that are too weak for decision making:
"A_{25} will get me there on time if there's no accident on the bridge and it doesn't rain and my tires remain intact etc etc."

Probability

Probabilistic assertions summarize effects of
- laziness: failure to enumerate exceptions, qualifications, etc.
- ignorance: lack of relevant facts, initial conditions, etc.

Subjective or Bayesian probability:
Probabilities relate propositions to one's own state of knowledge
- e.g., $P(A_{25} | \text{no reported accidents}) = 0.06$

These are not claims of a "probabilistic tendency" in the current situation (but might be learned from past experience of similar situations)

Probabilities of propositions change with new evidence:
- e.g., $P(A_{25} | \text{no reported accidents, 5 a.m.}) = 0.15$

(Discrete) random variables

What is a random variable?

Discrete random variable, X, can take on many (possibly infinite) values, called the *state space* or *domain* A={1,2,3,4,5,6} (e.g., a die)

$$a \in \{1, 2, 3, 4, 5, 6\} = A$$

a is a *random variable*

this is the *domain* of *a*

Another example:

Suppose *b* denotes whether it is raining or clear outside:

$$b \in \{rain, clear\} = B$$

Probability distribution

A probability distribution associates each with a probability of occurrence, represented by a *probability mass function (pmf)*.

A probability table is one way to encode the distribution:

$$a \in \{1, 2, 3, 4, 5, 6\} = A \qquad b \in \{rain, clear\} = B$$

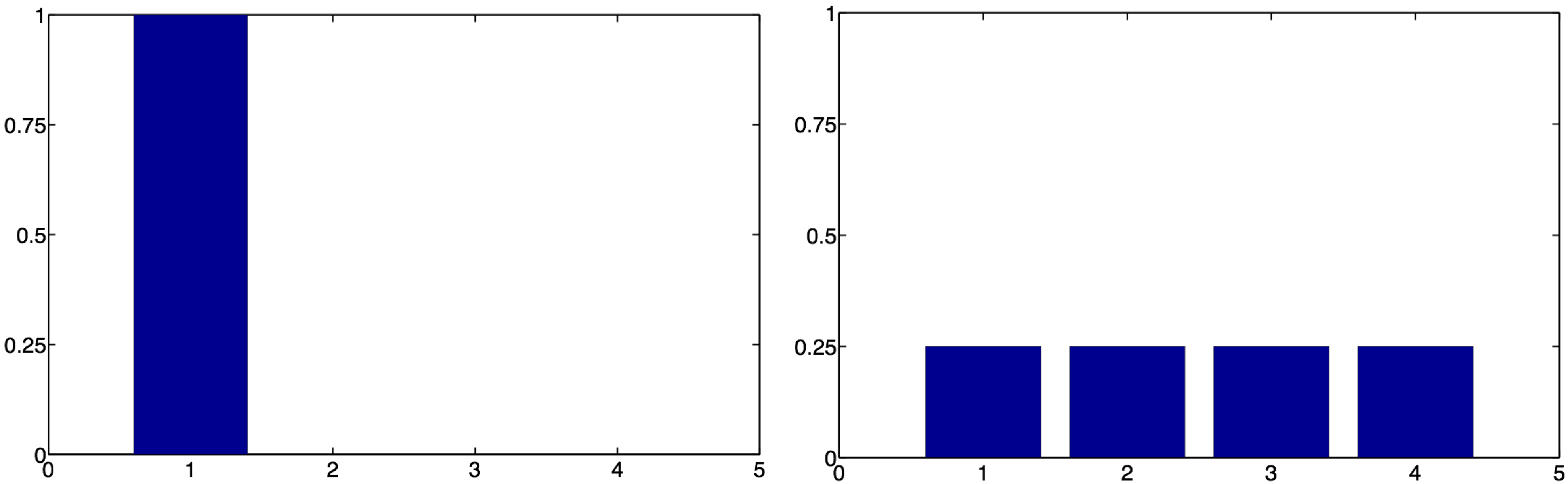
a	P(a)
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

b	P(b)
rain	1/4
clear	3/4

<u>All probability distributions must satisfy the following:</u>

1. $\forall a \in A, a \geq 0$
2. $\sum_{a \in A} a = 1$

Example pmfs



Two pmfs over a state space of $X = \{1,2,3,4\}$

Writing probabilities

a	P(a)
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

b	P(b)
rain	1/4
clear	3/4

For example: $p(a = 2) = 1/6$

$$p(b = clear) = 3/4$$

But, sometimes we will abbreviate this as: $p(2) = 1/6$

$$p(clear) = 3/4$$

Types of random variables

Propositional or Boolean random variables

- e.g., *Cavity* (do I have a cavity?)
- *Cavity* = *true* is a proposition, also written *cavity*

Discrete random variables (finite or infinite)

- e.g., *Weather* is one of ⟨*sunny, rain, cloudy, snow*⟩
- *Weather* = *rain* is a proposition
- Values must be exhaustive and mutually exclusive

Continuous random variables (bounded or unbounded)

- e.g., Temp < 22.0

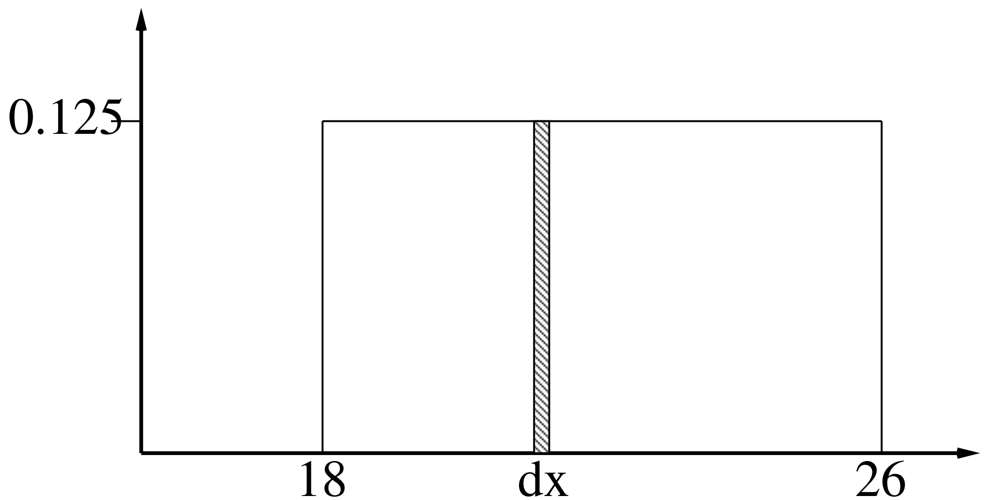
Continuous random variables

Cumulate distribution function (*cdf*), *F*(*q*)=(*X*<*q*) with *P*(*a*<*X*≤*b*)=*F*(*b*)-*F*(*a*)

Probability density function (*pdf*), $f(x) = \frac{d}{dx} F(x)$ with $P(a < X \leq b) = \int_a^b f(x)$

Express distribution as a parameterized function of value:

- e.g., *P*(*X* = *x*) = *U*[18, 26](*x*) = uniform density between 18 and 26



Here *P* is a density; integrates to 1.

P(*X* = 20.5) = 0.125 really means

$$\lim_{dx \to 0} P(20.5 \leq X \leq 20.5 + dx) / dx = 0.125$$

Joint probability distributions

Given random variables: $X_1, X_2, \ldots, X_n$

The *joint distribution* is a probability assignment to all combinations: $P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$

or: $P(x_1, x_2, \ldots, x_n)$

Sometimes written as: $P(X_1 = x_1 \wedge X_2 = x_2 \wedge \ldots \wedge X_n = x_n)$

As with single-variate distributions, joint distributions must satisfy:

1. $P(x_1, x_2, \ldots, x_n) \geq 0$

2. $\sum_{x_1, \ldots, x_n} P(x_1, x_2, \ldots, x_n) = 1$

Prior or unconditional probabilities of propositions
e.g., P (*Cavity* = *true*) = 0.1 and P (*Weather* = *sunny*) = 0.72
correspond to belief prior to arrival of any (new) evidence

Joint probability distributions

Joint distributions are typically written in table form:

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

How many entries do I need here?

Marginalization

Given P(T,W), calculate P(T) or P(W)...

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(T) = \sum_{w \in W} P(T, w)$$

T	P(T)
hot	0.5
cold	0.5

$$P(W) = \sum_{t \in T} P(t, W)$$

W	P(W)
sun	0.5
rain	0.4

Marginalization

$P(X,Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

$$P(x) = \sum_y P(x,y)$$

$$P(y) = \sum_x P(x,y)$$

$P(X)$

X	P
+x	
-x	

$P(Y)$

Y	P
+y	
-y	

Conditional Probabilities

Conditional or posterior probabilities

- e.g., $P(cavity|toothache) = 0.8$
- i.e., given that toothache is all I know

Notation for conditional distributions: $\mathbf{P}(Cavity|Toothache)$ = 2-element vector of 2-element vectors

If we know more, e.g., cavity is also given, then we have $P(cavity|toothache, cavity) = 1$

- Note: the less specific belief remains valid after more evidence arrives, but is not always useful

New evidence may be irrelevant, allowing simplification

- e.g., $P(cavity|toothache, redsoxwin) = P(cavity|toothache) = 0.8$

This kind of inference, sanctioned by domain knowledge, is crucial

Conditional Probabilities

Conditional probability: $P(A \mid B) = \dfrac{P(A,B)}{P(B)}$ (if P(B)>0)

Example: Medical diagnosis

Product rule: P(A,B) = P(A $\wedge$ B) = P(A|B)P(B)

Marginalization with conditional probabilities:

$$P(A) = \sum_{b \in B} P(A \mid B = b) P(B = b)$$

This formula/rule is called the *law of of total probability*

Chain rule is derived by successive application of product rule:
$P(X_1,\ldots,X_n) = P(X_1,\ldots,X_{n-1})\, P(X_n|X_1,\ldots,X_{n-1})$
$= P(X_1,\ldots,X_{n-2})\, P(X_{n-1}|X_1,\ldots,X_{n-2})\, P(X_n|X_1,\ldots,X_{n-1}) = \ldots$
$= \Pi^n_{i=1} P(X_i|X_1,\ldots,X_{i-1})$

Conditional Probabilities

$P(sun|hot) \equiv$ Probability that it is sunny *given* that it is hot.

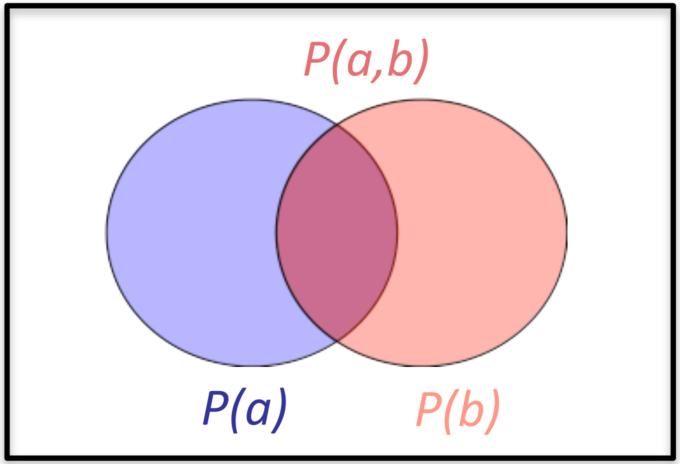
T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Conditional Probabilities

Calculate the conditional probability using the product rule:

Product rule

$$P(a|b) = \frac{P(a,b)}{P(b)}$$



T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)} = \frac{0.2}{0.5} = 0.4$$

$$= P(W=s,T=c) + P(W=r,T=c)$$

$$= 0.2 + 0.3 = 0.5$$

Conditional Probabilities

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- P(+x | +y) ?
- P(-x | +y) ?
- P(-y | +x) ?

Conditional distribution

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

W	$\mathrm{P}(\mathrm{W} \mid t = hot)$
sun	0.8
rain	0.2

$$P(W|t) = \frac{P(W,t)}{P(t)}$$

Conditional distribution

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

W	P(W $\mid t = hot$)
sun	0.8
rain	0.2

$$P(W|t) = \frac{P(W,t)}{P(t)}$$

$$P(sun|hot) = \frac{P(sun,hot)}{P(hot)} = \frac{P(sun,hot)}{P(sun,hot) + P(rain,hot)}$$

$$= \frac{0.4}{0.4 + 0.1}$$

Conditional distribution

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W|t) = \frac{P(W,t)}{P(t)}$$

W	P(W | $t = hot$)
sun	0.8
rain	0.2

W	P(W | $t = cold$)
sun	0.4
rain	0.6

Conditional distribution

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W|t) = \frac{P(W,t)}{P(t)}$$

W	P(W $\mid t = hot$)
sun	0.8
rain	0.2

W	P(W $\mid t = cold$)
sun	0.4
rain	0.6

$$P(sun|cold) = \frac{P(sun, cold)}{P(cold)} = \frac{P(sun, cold)}{P(sun, cold) + P(rain, cold)}$$

$$= \frac{0.2}{0.2 + 0.3}$$

Normalization

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)

Can we avoid explicitly computing this?

$$\ldots t) = \frac{P(W,t)}{P(t)}$$

W	P(W $\mid t = hot$)
sun	0.8
rain	0.2

W	P(W $\mid t = cold$)
sun	0.4
rain	0.6

$$P(sun|cold) = \frac{P(sun,cold)}{P(cold)} = \frac{P(sun,cold)}{P(sun,cold)+P(rain,cold)}$$

$$= \frac{0.2}{0.2+0.3}$$

Normalization

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

W	P(W, t=hot)
sun	0.4
rain	0.1

W	$\mathrm{P}(\mathrm{W} \mid t = hot)$
sun	0.8
rain	0.2

Select corresponding elements from the joint distribution

Scale the numbers so that they sum to 1

$$P(sun|cold) = \frac{P(sun, cold)}{P(cold)} = \frac{P(sun, cold)}{P(sun, cold) + P(rain, cold)}$$

Normalization

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

→

W	P(W, t=hot)
sun	0.4
rain	0.1

→

W	$P(W \mid t = hot)$
sun	0.8
rain	0.2

Select corresponding elts from the joint distribution

Scale the numbers so that they sum to 1.

$$P(sun|cold) = \frac{P(sun, cold)}{P(cold)} = \frac{P(sun, cold)}{P(sun, cold) + P(rain, cold)}$$

The only purpose of this denominator is to make the distribution sum to one.
– we achieve the same thing by scaling.

Normalization

P(X | Y=-y) ?

$P(X, Y)$

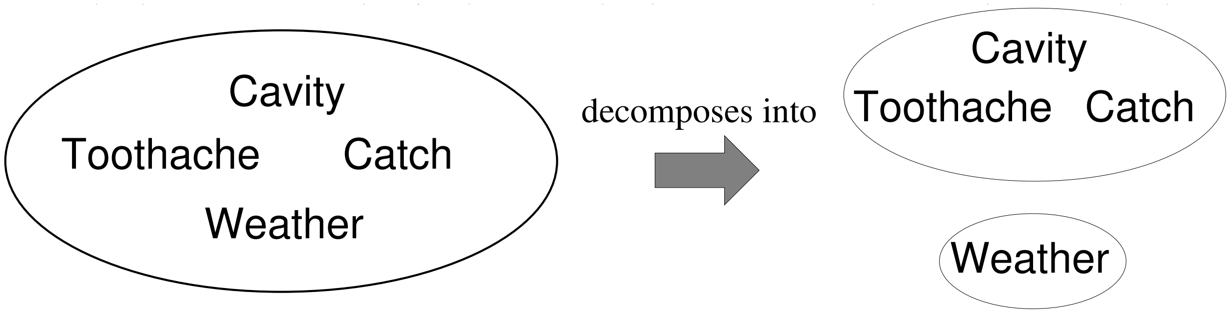
X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

→ ? → ?

Independence

A and *B* are *independent* iff
$P(A|B) = P(A)$ or $P(B|A) = P(B)$ or $P(A,B) = P(A)P(B)$

$P(Toothache, Catch, Cavity, Weather)$
$= P(Toothache, Catch, Cavity)P(Weather)$



32 entries reduced to 12; for *n* independent biased coins, $2^n \rightarrow n$

Absolute independence powerful but rare

Dentistry is a large field with hundreds of variables, none of which are independent. What to do?

Conditional independence

P(Toothache, Cavity, Catch) has 23 − 1 = 7 independent entries

If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:

(1)P(*catch*|*toothache, cavity*) = *P*(*catch*|*cavity*)

The same independence holds if I haven't got a cavity:

(2) P(*catch*|*toothache*, ¬*cavity*) = P(*catch*|¬*cavity*)

Catch is conditionally independent of *Toothache* given *Cavity*:

P(*Catch*|*Toothache*, *Cavity*) = P(*Catch*|*Cavity*)

Equivalent statements:

P(*Toothache*|*Catch, Cavity*)=P(*Toothache*|*Cavity*)

P(*Toothache*, *Catch*|*Cavity*)=P(*Toothache*|*Cavity*)P(*Catch*|*Cavity*)

Conditional independence

Write out full joint distribution using chain rule:

P(*Toothache, Catch, Cavity*)

= P(*Toothache*|*Catch, Cavity*)P(*Catch, Cavity*)

= P(*Toothach*e|*Catch, Cavity*)P(*Catch*|*Cavity*)P(*Cavity*)

= P(*Toothache*|*Cavity*)P(*Catch*|*Cavity*)P(*Cavity*)

2 + 2 + 1 = 5 independent numbers (equations 1 and 2 remove 2)

In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in *n* to linear in *n*.

Conditional independence is our most basic and robust form of knowledge about uncertain environments.

Bayes' Rule

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

Thomas Bayes?

Bayes' Rule

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

It's easy to derive from the product rule:

$$P(a,b) = P(b|a)P(a) = \underbrace{P(a|b)}P(b)$$

Solve for this

Using Bayes' Rule

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

Using Bayes' Rule

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

But harder to estimate this

It's often easier to estimate this

Bayes' Rule Example

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

Suppose you have a stiff neck...

Suppose there is a 70% chance of meningitis if you have a stiff neck:

stiff neck → s, meningitis → m

$$P(s|m) = 0.7$$

What are the chances that you have meningitis?

Bayes' Rule Example

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

Suppose you have a stiff neck...

Suppose there is a 70% chance of meningitis if you have a stiff neck:

stiff neck → s, meningitis → m

$$P(s|m) = 0.7$$

What are the chances that you have meningitis?

We need a little more information...

Bayes' Rule Example

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

$$P(s|m) = 0.7$$

$$P(s) = 0.01$$ ← Prior probability of stiff neck

$$P(m) = \frac{1}{50000}$$ ← Prior probability of meningitis

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.7 \times \frac{1}{50000}}{0.01} = 0.0014$$

Bayes' Rule Example

- Given:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

- What is $P(W|dry)$?

Bayes' rule and conditional independence

$\mathbf{P}(Cavity|toothache, catch)$
$= \alpha\mathbf{P}(toothache, catch|Cavity)\mathbf{P}(Cavity)$
$= \alpha\mathbf{P}(toothache|Cavity)\mathbf{P}(catch|Cavity)\mathbf{P}(Cavity)$

This is an example of a naive Bayes model:
$P(Cause, Effect_1, \ldots, Effect_n) = P(Cause)\Pi_i \mathbf{P}(Effect_i|Cause)$

Total number of parameters is linear in n

Making decisions under uncertainty

Suppose I believe the following:

$P(A_{25}$ gets me there on time$|\ldots) = 0.04$

$P(A_{90}$ gets me there on time$|\ldots) = 0.70$

$P(A_{120}$ gets me there on time$|\ldots) = 0.95$

$P(A_{1440}$ gets me there on time$|\ldots) = 0.9999$

Which action to choose?

Making decisions under uncertainty

Suppose I believe the following:

$P(A_{25}$ gets me there on time$|...) = 0.04$

$P(A_{90}$ gets me there on time$|...) = 0.70$

$P(A_{120}$ gets me there on time$|...) = 0.95$

$P(A_{1440}$ gets me there on time$|...) = 0.9999$

Which action to choose?

Depends on my preferences for missing flight vs. airport cuisine, etc.

Utility theory is used to represent and infer preferences

Decision theory = utility theory + probability theory

Making decisions under uncertainty

Rational decision making requires reasoning about one's *uncertainty* and *objectives*

Previous section focused on uncertainty

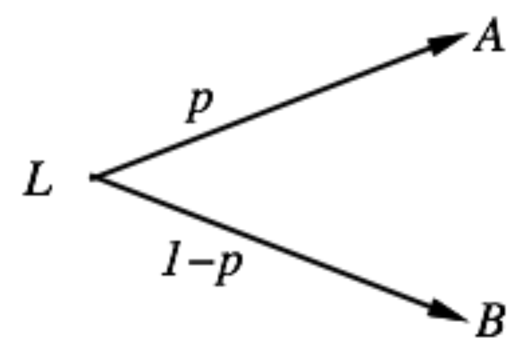
This section will discuss how to make rational decisions based on a *probabilistic model* and *utility function*

Focus will be on single step decisions, next week we will consider sequential decision problems

Preferences

An agent chooses among prizes (A, B, etc.) and lotteries, i.e., situations with uncertain prizes

Lottery $L=[p, A;\ (1-p), B]$



Notation:

$A \succ B$ A preferred to B

$A \sim B$ indifference between A and B

$A \succsim B$ B not preferred to A

Rational preferences

Idea: preferences of a rational agent (*not* a human!) must obey constraints

Rational preferences $\Rightarrow$ behavior describable as maximization of expected utility

The Axioms of Rationality:

Orderability

$$(A \succ B) \vee (B \succ A) \vee (A \sim B)$$

Transitivity

$$(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$$

Continuity

$$A \succ B \succ C \Rightarrow \exists p \; [p, A; \; 1-p, C] \sim B$$

Substitutability

$$A \sim B \Rightarrow [p, A; \; 1-p, C] \sim [p, B; 1-p, C]$$

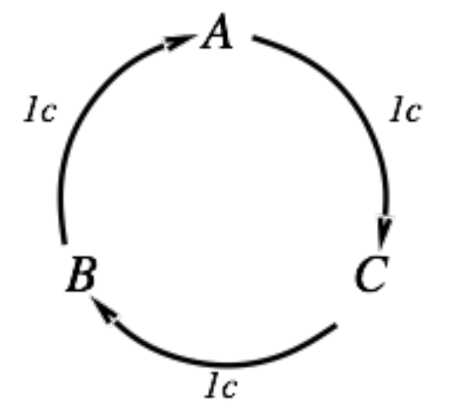
Monotonicity

$$A \succ B \Rightarrow$$
$$(p \geq q \Leftrightarrow [p, A; \; 1-p, B] \succsim [q, A; \; 1-q, B])$$

Rational preferences

Violating the constraints leads to self-evident irrationality

For example: an agent with intransitive preferences can be induced to give away all its money



If $B > C$, then an agent who has C would pay (say) 1 cent to get B

If $A > B$, then an agent who has B would pay (say) 1 cent to get A

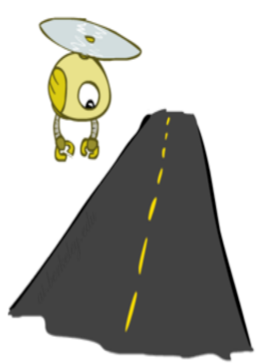
If $C > A$, then an agent who has A would pay (say) 1 cent to get C

Reminder: Expectations

The expected value of a function of a random variable is the average, weighted by the probability distribution over outcomes

Example: How long to get to the airport?

Time:	20 min		30 min		60 min		
	x	+	x	+	x	⇨	35 min
Probability:	0.25		0.50		0.25		



Mean and variance

Mean, μ, or expected value:

Discrete: $$\mathbb{E}[X] = \sum_{x \in X} x P(x)$$

Continuous: $$\mathbb{E}[X] = \int_x x P(x) dx$$

Variance: $$\mathrm{var}[X] = \mathbb{E}[(X-\mu)^2]$$

Maximizing expected utility (MEU)

Theorem (Ramsey, 1931; von Neumann and Morgenstern, 1944): Given preferences satisfying the constraints there exists a real-valued function U such that

$$U(A) \geq U(B) \Leftrightarrow A \succsim B$$

$$U(A) > U(B) \Leftrightarrow A \succ B$$

$$U(A) = U(B) \Leftrightarrow A \sim B$$

$$U([p_1, s_1; \ldots; p_n, s_n]) = \sum_i p_i U(s_i)$$

MEU principle: Choose the action that maximizes expected utility

Preferences lead to utilities

Note: an agent can be entirely rational (consistent with MEU) without ever representing or manipulating utilities and probabilities

E.g., a lookup table for perfect tic-tac-toe

Although a utility function must exist, it is not unique

If $U'(S)=aU(S)+b$ and a and b are constants with $a>0$, then preferences of U' are the same as U

E.g., temperatures in Celcius, Fahrenheit, Kelvin

MEU continued

Agent has made some (imperfect) observation o of the state of the world

If the agent executes action a, the probability the state of the world becomes s' is given by $P(s' \mid o, a)$

Preferences on outcomes is encoded using utility function $U(s)$

Expected utility: $$EU(a \mid o) = \sum_{s'} P(s' \mid a, o) U(s')$$

Principal of maximum expected utility says that a rational agent should choose the action that maximizes expected utility $a^* = argmax_a EU(a \mid o)$

Utilities: preference elicitation

When building a decision-making or decision-support system, it is often helpful to infer the utility function from a human

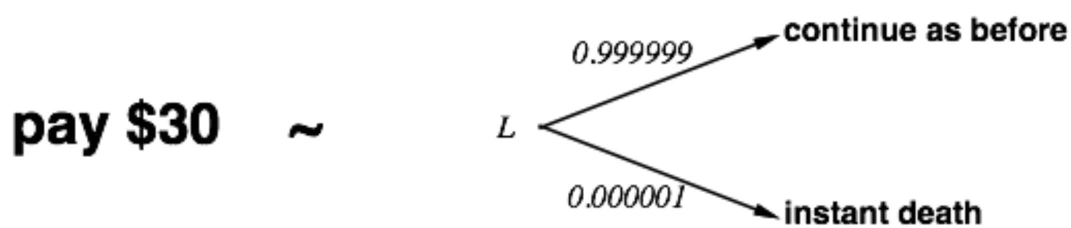
Utilities map states to real numbers. Which numbers?

Standard approach to assessment of human utilities: compare a given state A to a standard lottery L_p that has

"best possible prize" $u_\top$ with probability p

"worst possible catastrophe" $u_\perp$ with probability $(1 - p)$

Adjust lottery probability p until $A \sim L_p$



Alternatively, set best possible utility to 1 and worst possible to 0

Utility scales

Normalized utilities: $u_\top = 1.0$, $u_\bot = 0.0$

Micromorts: one-millionth chance of death

Useful for Russian roulette, paying to reduce product risks, etc.

QALYs: quality-adjusted life years

Useful for medical decisions involving substantial risk

Note: behavior is invariant w.r.t. positive linear transformation

$$U'(x) = k_1 U(x) + k_2 \quad \text{where } k_1 > 0$$

With deterministic prizes only (no lottery choices), only ordinal utility can be determined, i.e., total order on prizes

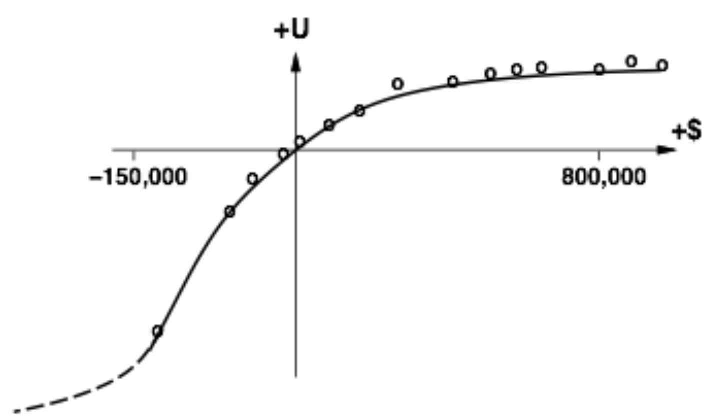
Money

Money does not behave as a utility function

Given a lottery L with expected monetary value $EMV(L)$, usually $U(L) < U(EMV(L))$, i.e., people are risk-averse

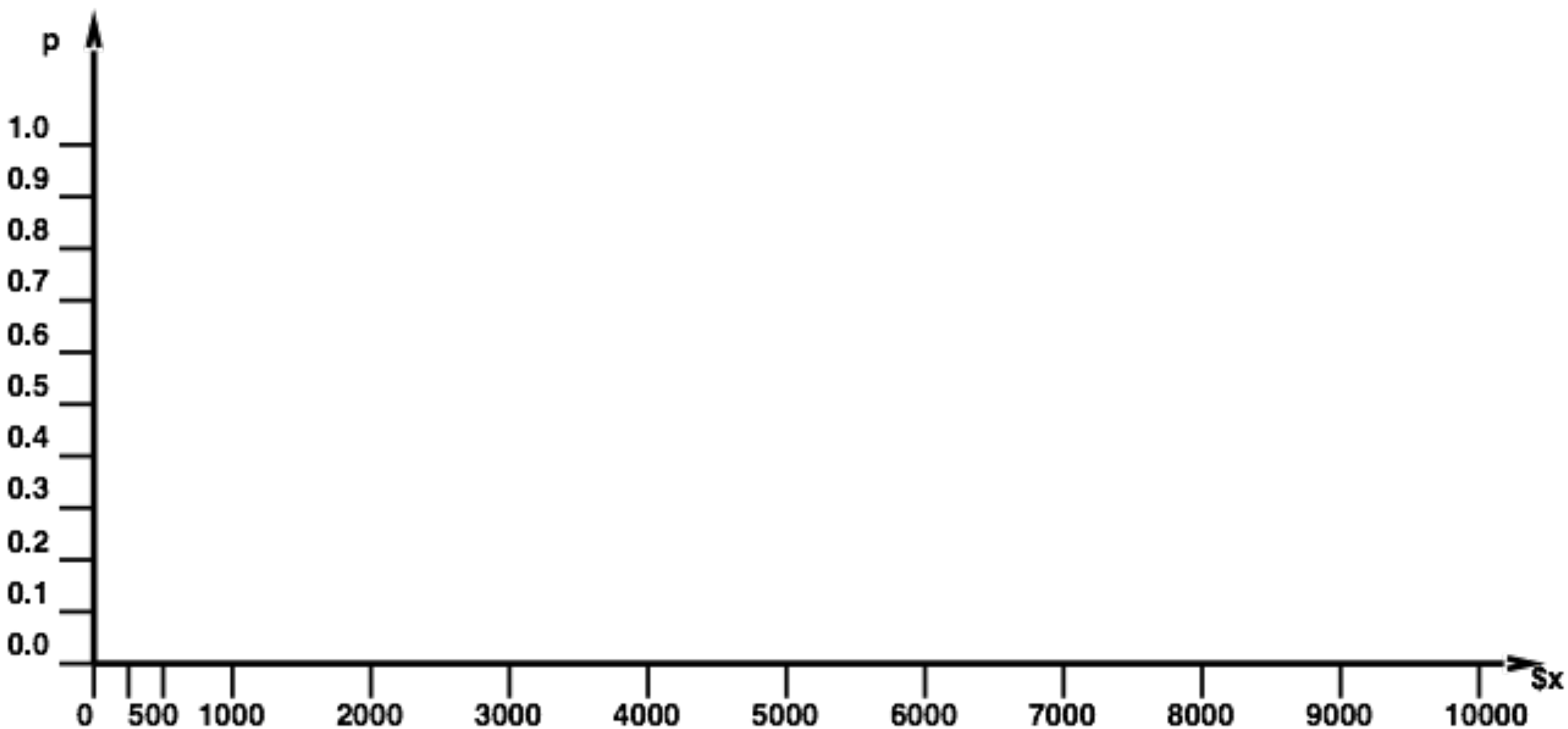
Utility curve: for what probability p am I indifferent between a prize x and a lottery $[p, \$M; (1-p), \$0]$ for large M?

Typical empirical data, extrapolated with risk-prone behavior (utility of money is proportional to the logarithm of the amount):



Student group utility

Who prefers the lottery at different values of p? (M=10,000)



Summary

Probability is a rigorous formalism for uncertain knowledge

Joint probability distribution specifies probability of every atomic event

Queries can be answered by summing over atomic events

For nontrivial domains, we must find a way to reduce the joint size

Independence and conditional independence provide the tools

Next time: sequential decision making!